THE MEAN-VALUE ITERATION FOR SET-VALUED MAPPINGS

PETER K. F. KUHFITTIG

Abstract. In this note Krasnoselskii's iteration procedure

\[ x_{n+1} = \frac{1}{2} (I + T)x_n \]

is extended to certain classes of set-valued mappings.

1. Introduction. Let \( C \) be a convex subset of a Banach space \( B \) and \( T \) a self-mapping of \( C \) and consider the following iteration process of a type introduced by Mann [7]: for an arbitrary starting point \( x_0 \in C \)

\[ x_{n+1} = (1 - c_n)x_n + c_nTx_n, \quad n = 0, 1, 2, \ldots, \quad (*) \]

where \( c_n \in [a, b] \) for \( 0 < a < b < 1 \). The special case \( c_n = \frac{1}{2} \) for all \( n \) was first introduced by Krasnoselskii [5], who showed that the sequence converges to a fixed point of \( T \) if \( T \) is nonexpansive, \( B \) uniformly convex, and \( C \) compact. This result remains valid if \( c_n = \alpha, 0 < \alpha < 1 \) (Schaefer [12]). Moreover, it is sufficient to assume that \( B \) is strictly convex (Edelstein [3]). Retaining uniform convexity, Browder and Petryshyn [1] assumed \( C \) to be closed and \( T \) demicompact. Under the latter conditions, the sequence \((*)\) converges to a fixed point of \( T \) if \( T \) is merely continuous and quasinonexpansive, that is, nonexpansive about its set of fixed points, assumed nonempty. (See Corollary which follows.) The iteration \((*)\) has been investigated by Senter and Dotson [13].

In this paper we shall consider an analogous iteration for a mapping \( T: C \to K(C) \), where \( K(C) \) is the family of nonempty compact subsets of \( C \). It is assumed that one fixed point \( z \) is known and that \( T \) is nonexpansive about this point, that is, for all \( x \in C \)

\[ D(Tx, Tz) < \| x - z \|, \]

where \( D \) is the Hausdorff metric on \( K(C) \). The iteration procedure is designed to generate additional fixed points.

Regarding the existence of fixed points, it was shown by Lim [6] that if \( C \) is a convex closed and bounded subset of a uniformly convex Banach space, then a nonexpansive mapping \( T: C \to K(C) \) has a fixed point. This result has recently been extended by Downing and Kirk [2].

2. The sequence. Let \( z \in Tz \) be the known fixed point. Since \( Tx \) is compact and \( D \) the Hausdorff metric, we can find for every \( x \in C \) a point \( p_x \in Tx \) such that

\[ \| z - p_x \| < D(Tz, Tx). \]
Using this information, suppose we construct a sequence \( \{x_n\} \) in \( C \) as follows: let \( x_0 \in C \) and \( p_0 \in Tx_0 \). Next let
\[
x_1 = (1 - c_0)x_0 + c_0 p_0,
\]
where \( c_0 \in [a, b] \), \( 0 < a < b < 1 \). Then we can find \( p_1 \in Tx_1 \) such that
\[
\|z - p_1\| < D(Tz, Tx_1)
\]
by the prior comments. Now let
\[
x_2 = (1 - c_1)x_1 + c_1 p_1.
\]
Since \( Tx_2 \) is compact, we can find \( p_2 \in Tx_2 \) such that
\[
\|z - p_2\| < D(Tz, Tx_2).
\]
Continuing in this manner
\[
x_{n+1} = (1 - c_n)x_n + c_n p_n, \quad n = 0, 1, 2, \ldots,
\]
where \( c_n \in [a, b] \) for \( 0 < a < b < 1 \), \( p_n \in Tx_n \), and
\[
\|z - p_n\| < D(Tz, Tx_n).
\]
Since \( T \) is not even assumed to be quasinonexpansive, we do require continuity in the following sense.

**Definition 1.** A mapping \( T: C \to K(C) \) is continuous if for any sequence \( \{y_n\} \) in \( C \), \( y_n \to y \) implies that \( Ty_n \to Ty \).

**Definition 2 (Petryshyn [10]).** A mapping \( U: C \to B \) of a subset \( C \) of a Banach space \( B \) into \( B \) is said to be demicompact if whenever \( \{x_n\} \subset C \) is a bounded sequence and \( \{x_n - Ux_n\} \) is a convergent sequence, then there exists a subsequence \( \{x_{n_k}\} \) which is convergent.

For set-valued mappings we have the following analogous definition.

**Definition 3.** A mapping \( T: C \to K(C) \) will be called demicompact if whenever \( \{x_n\} \subset C \) is a bounded sequence and \( \{\text{dist}(x_n, Tx_n)\} \) is a convergent sequence, then there is a subsequence \( \{x_{n_k}\} \) which is convergent.

In the proof of the first theorem we are going to need the following two lemmas.

**Lemma 1 (Schaefer [12]).** Let \( B \) be a uniformly convex Banach space. Then for \( \varepsilon > 0 \), \( d > 0 \), and \( \alpha \in (0, 1) \) the inequalities \( \|x\| < d \), \( \|y\| < d \), and \( \|x - y\| > \varepsilon \) imply that
\[
\|(1 - \alpha)x + \alpha y\| < \left[ 1 - 2 \delta(\varepsilon/d) \min(1 - \alpha, \alpha) \right] d;
\]
\( \delta \) is strictly increasing.

**Lemma 2 (Nadler [9]).** Let \( \{A_n\} \) be a sequence of sets in \( K(C) \) and suppose
\[
\lim_{n \to \infty} D(A_n, A_0) = 0, \quad \text{where } A_0 \in K(C).
\]
Then if \( x_n \in A_n \), \( n = 1, 2, \ldots \), and if \( \lim_{n \to \infty} x_n = x_0 \) it follows that \( x_0 \in A_0 \).

3. Results.

**Theorem 1.** Let \( C \) be a nonempty convex closed subset of a uniformly convex Banach space \( B \). If \( T: C \to K(C) \) is a continuous demicompact mapping which is nonexpansive about a known fixed point \( z \), then for the sequence \( \{x_n\} \) defined previously, (a) there exists a subsequence \( \{x_{n_k}\} \) converging to a fixed point of \( T \) and
(b) every cluster point of \( \{x_n\} \) is a fixed point of \( T \). (In particular, every convergent subsequence of \( \{x_n\} \) converges to a fixed point.)

**Proof.** The first step is to show that for the sequence \( \{x_n\} \) constructed previously

\[
\|x_n - p_n\| \to 0 \quad \text{as} \quad n \to \infty.
\]

If not, then there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) and a number \( \epsilon > 0 \) such that

\[
\|x_{n_k} - p_{n_k}\| \geq \epsilon. \quad (1)
\]

Since \( p_n \in Tx_n \),

\[
\|z - p_n\| \leq D(Tz, Tx_n) < \|z - x_n\|. \quad (2)
\]

Then by (1), (2) and Lemma 1 there exists

\[
\delta = \delta(\epsilon/\|z - x_{n_k}\|) > 0
\]

such that

\[
\|z - x_{n_{k+1}}\| = \|z - (1 - c_{n_k})x_{n_k} - c_{n_k}p_{n_k}\|
\]

\[
= \|(1 - c_{n_k})(z - x_{n_k}) + c_{n_k}(z - p_{n_k})\|
\]

\[
< (1 - \delta\gamma)\|z - x_{n_k}\|,
\]

where \( \gamma = 2 \min(1 - c_{n_k}, c_{n_k}) \). From

\[
\|z - x_{n_{k+1}}\| = \|(1 - c_{n_k})(z - x_{n_k}) + c_{n_k}(z - p_{n_k})\| < \|z - x_{n_k}\|, \quad (3)
\]

the sequence \( \{\|z - x_n\|\} \) is nonincreasing, and since \( \delta \) is strictly increasing, the sequence

\[
\{\delta(\epsilon/\|z - x_{n_k}\|)\}
\]

in nondecreasing. Since we also have

\[
\|z - x_{n_k}\| < \|z - x_{n_{k+1}}\| < (1 - \delta\gamma)\|z - x_{n_{k+1}}\|
\]

for

\[
\delta = \delta(\epsilon/\|z - x_{n_{k+1}}\|)
\]

and

\[
\gamma = 2 \min(1 - c_{n_k}, c_{n_k}),
\]

it follows that

\[
\|z - x_{n_k}\| \to 0 \quad \text{as} \quad j \to \infty.
\]

By (2) \( \|z - p_n\| \to 0 \), whence \( \|x_{n_k} - p_{n_k}\| \to 0 \) as \( j \to \infty \), contradicting statement (1). Hence

\[
\|x_n - p_n\| \to 0 \quad \text{as} \quad n \to \infty, \quad (4)
\]

which was to be shown.

It now follows from (4) that \( \text{dist}(x_n, Tx_n) \to 0 \) as \( n \to \infty \). Moreover, by (3), \( \{x_n\} \) is a bounded sequence. So by demicompactness there exists a subsequence \( \{x_{n_k}\} \) of
\{x_n\} such that \(x_n \to z_0 \in C\). Also, from
\[
\|p_n - z_0\| \leq \|p_n - x_n\| + \|x_n - z_0\|,
\]
we have that \(p_n \to z_0\). But \(Tx_n \to Tz_0\) by continuity. Consequently, since \(p_n \in Tx_n\), \(z_0 \in Tz_0\) by Lemma 2.

Finally, if \(w_0\) is a cluster point of \(\{x_n\}\), there exists a subsequence converging to \(w_0\), which is a fixed point by the above argument. This completes the proof.

Recall from §2 that \(p_x \in Tx\) is a point for which \(\|z - p_x\| < D(Tz, Tx)\). Suppose for every such \(p_x \in Tx\) and \(p_y \in Ty\), \(T : C \to K(C)\) satisfies the condition
\[
D(Tx, Ty) < \alpha \|x - p_x\| + \beta \|y - p_y\|
\]
for all \(x, y \in C\) and \(\alpha, \beta \in [0, \infty)\).

Then if \(\alpha = \beta \in [0, \frac{1}{2}]\) and if \(T\) is a point-to-point mapping, \(T\) is a Kannan mapping, first introduced by Kannan [4]. Such mappings have been studied extensively.

If \(p_w\) is chosen (without reference to the fixed point \(z\)) so that \(\|w - p_w\| = \text{dist}(w, Tw)\) and if \(\alpha = \beta = \frac{1}{2}\), then \(T\) becomes a set-valued Kannan mapping. Such mappings were studied by Shiau, et al. [14], [15]. Clearly, any Kannan mapping is of Type A.

**Theorem 2.** If \(T\) in Theorem 1 is of Type A, then \(Tx_n \to Tz_0\), where \(z_0 \in Tz_0\).

**Proof.** For every \(e > 0\) there exists \(N > 0\) such that
\[
D(Tx_n, Tx_m) < \alpha \|x_n - p_n\| + \beta \|x_m - p_m\| < e
\]
for all \(m, n > N\), so that \(\{Tx_n\}\) is a Cauchy sequence. Since \(C\) is complete, \((K(C), D)\) is complete (Michael [8]). Hence \(Tx_n \to L \in K(C)\). Since \(x_n \to z_0\), \(Tx_n \to Tz_0\), so that \(L = Tz_0\).

The result of Theorem 2 is, in one sense, the best possible. For if \(T\) is a nonexpansive set-valued mapping, then the natural analogue of Krasnoselskii’s procedure is the following: let \(x_0 \in C, q_0 \in Tx_0,\) and \(x_1 = \frac{1}{2}x_0 + \frac{1}{2}q_0\). Now choose \(q_1 \in Tx_1\) such that
\[
\|q_1 - q_0\| < D(Tx_1, Tx_0).
\]
In general, \(x_{n+1} = \frac{1}{2}x_n + \frac{1}{2}q_n\), where \(q_n \in Tx_n\) and
\[
\|q_n - q_{n-1}\| < D(Tx_n, Tx_{n-1}),
\]
whence
\[
\|q_n - q_{n-1}\| < \|x_n - x_{n-1}\|.
\]
This construction fails, however, as can be seen from the mapping \(T : R \to K(R)\) defined by \(Tx = [x - 1, x + 1]\). If for \(Tx_n = [x_n - 1, x_n + 1]\), we choose \(q_n = x_n + 1\), then the resulting sequence has no convergent subsequence.

If \(T\) is a point-to-point mapping, then \(q_n = Tx_n\), and \(\{x_n\}\) reduces to Krasnoselskii’s iteration. Now suppose \(T\) is continuous, demicompact, and quasinonexpansive with a nonempty set \(F\) of fixed points. Returning to the sequence (\(\ast\)), if \(z \in F\), then
\[
\|z - Tx_n\| < \|z - x_n\|.
\]
and, by the proof of Theorem 1, there exists a subsequence \( \{x_n\} \) for which \( x_n \to z_0 = Tz_0 \). Since \( \{\|z_0 - x_n\|\} \) is clearly nonincreasing, \( x_n \to z_0 \). This proves the following

**Corollary.** Let \( C \) be a nonempty convex closed subset of a uniformly convex Banach space. If \( T: C \to C \) is a continuous demicompact quasinonexpansive mapping with a nonempty set of fixed points, then the sequence (*) converges to a fixed point of \( T \).

This result is similar to Theorem 1.1' in [11] and Theorem 2 in [13].

**References**