AN IMPROVED ESTIMATE FOR THE BLOCH NORM OF FUNCTIONS IN DOOB’S CLASS

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Abstract. For any fixed $0 < \rho < 2\pi$, $\mathcal{D}(\rho)$ is the family of all holomorphic functions in $\Delta$ which satisfy (i) $f(0) = 0$, and (ii) $\lim_{|z| \to \rho} |f(z)| > 1$, for all $\tau$ lying on some arc $\Gamma_\tau \subseteq \partial \Delta$ with arclength $|\Gamma_\tau| > \rho$. We showed that for each $f \in \mathcal{D}(\rho)$ there exists a point $z_f \in \Delta$ at which

$$|f'(z_f)(1 - |z_f|^2)| > \frac{2}{e} \frac{\sin(\pi - (\rho/2))}{(\pi - (\rho/2))}.$$

In this paper we improve this estimate by replacing the quantity $\pi - (\rho/2)$ with a value $\theta(\rho)$ which lies between 0 and $\pi - (\rho/2)$ and so improves the estimate. The value $\theta(\rho)$ is defined as the (unique) solution in this interval of the equation

$$F_\rho(\theta) = \log(\cot(\rho/4)\cot(\theta/2)) - \frac{\theta}{\sin \theta} = 0.$$

1. In a series of papers ([4]–[8]) J. L. Doob introduced a family $\mathcal{D}(\rho)$ of holomorphic functions in the unit disc $\Delta$, whose boundary we denote by $\partial \Delta$. For any fixed $0 < \rho < 2\pi$, $\mathcal{D}(\rho)$ is the family of all holomorphic functions in $\Delta$ which satisfy (i) $f(0) = 0$, and (ii) $\lim_{|z| \to \rho} |f(z)| > 1$, for all $\tau$ lying on some arc $\Gamma_\tau \subseteq \partial \Delta$ with arclength $|\Gamma_\tau| > \rho$. Doob posed the question as to whether the set of Bloch norms $\{\|f\|_B = \sup_{z \in \Delta} |f'(z)(1 - |z|^2)|\}_{f \in \mathcal{D}(\rho)}$ has a positive lower bound. We showed in [10] that for each $f \in \mathcal{D}(\rho)$ there exists a point $z_f \in \Delta$ at which

$$|f'(z_f)(1 - |z_f|^2)| > \frac{2}{e} \frac{\sin(\pi - (\rho/2))}{(\pi - (\rho/2))}.$$

In this paper we improve this estimate by replacing the quantity $\pi - (\rho/2)$ with a value $\theta(\rho)$ which lies between 0 and $\pi - (\rho/2)$. The value $\theta(\rho)$ is defined as the (unique) solution in this interval of the equation

$$F_\rho(\theta) = \log(\cot(\rho/4)\cot(\theta/2)) - \frac{\theta}{\sin \theta} = 0.$$

Functions in $\mathcal{D}(\rho)$ produce upper estimates to the various Bloch constants [2]. For $f$ holomorphic in $\Delta$ set $b(f) = \sup\{r|\text{there exists a domain }\Delta_1 \subseteq \Delta \text{ such that } f \text{ is univalent on } \Delta_1 \text{ and } f(\Delta_1) \text{ contains a disc of radius } r\}$. If $\mathcal{B}$ denotes the family of holomorphic functions in $\Delta$ normalized by $|f'(0)| > 1$, then the Bloch constant $B$ is defined as

$$B = \inf_b b(f), \quad f \in \mathcal{B}.$$
If $\mathbb{B}_s$ denotes the subfamily of $\mathbb{B}$ of all univalent functions then $B_s = \inf b(f)$, $f \in \mathbb{B}_s$. It is known that $\sqrt[3]{\frac{3}{4}} < B < 0.472$; $0.544 < B_s < 0.658$. These lower estimates are due to Heins [9] and Landau [11], respectively, while the upper estimates are due to Ahlfors and Grunsky [1] and R. Robinson [13], respectively. With no loss of generality the normalization $|f'(0)| > 1$ can be relaxed. If $L$ is a Möbius transformation of $\Delta$ onto $\Delta$ taking $0$ into $z$ then we have both $b(f \circ L) = b(f)$ and $(f \circ L)'(0) = (1 - |z|^2)|f'(z)|$. So we replace $\mathbb{B}$ (and $\mathbb{B}_s$) by replacing $|f'(0)| > 1$ by $|f'(z_f)|(1 - |z|^2) > 1$, for some $z_f \in \Delta$. The constants $B$ and $B_s$ remain unchanged. From our previous results as stated in (1.0) we see that $f \in \mathbb{B}(\rho)$ implies

$$
\left(\frac{e^{\frac{\pi - (\rho/2)}}{2 \sin(\pi - (\rho/2))}\right)f \in \mathbb{B},
$$

and so upper estimates for $B$ (and $B_s$) can be obtained from functions in $\mathbb{B}(\rho)$. Any improvement in the constant $(e/2)(\pi - (\rho/2))/\sin(\pi - (\rho/2))$ should be of some interest.

2. Main result. If $A \subseteq \Delta$, let $\partial A$ denote the topological boundary of $A$. If $\partial A \cap \partial A$ contains an arc $Y$ then, as usual, let $\omega(z, \Gamma, A)$ denote the harmonic measure at $z$ of $Y$ relative to $A$. If $A = \Delta$ then we define the lens-shaped domain in $\Delta$ by

$$
S(\alpha, \Gamma) = \{z \in \Delta|\omega(z, \Gamma, \Delta) > (\pi - \alpha)/\pi\}, \quad 0 < \alpha < \pi.
$$

It is easy to show that $\partial S(\alpha, \Gamma) \cap \Delta$ makes the angle $\alpha$ with $\partial A$. If $\alpha = \pi - (\rho/2)$ then $\partial S(\alpha, \Gamma)$ contains the origin. The proof of the main result is based upon a sharpened form of the Lehto-Virtanen differential two constant theorem [12] due to S. Dragosh and D. C. Rung [3]. For completeness we state a less general version suitable for our needs. It is similar to the version used in [10].

**Theorem D-R.** Let $f$ be meromorphic in $\Delta$. Fix a domain of the form $S(\alpha, \Gamma)$ and suppose

(i) $\sup_{z \in S(\alpha, \Gamma)} |f(z)| = M < \infty$;

(ii) there exists a point $q \in \partial S(\alpha, \Gamma) \cap \Delta$ at which $|f(q)| = M$;

(iii) for each $\tau \in \Gamma$, $\lim z \to \tau |f(z)| < m < M$.

Then

$$
|f'(q)|(1 - |q|^2) > \left(\frac{2 \sin \alpha}{\alpha}\right)M \log \frac{M}{m}. \quad (2.0)
$$

In preparation for the main result we need a few more details. For any fixed $0 < \rho < 2\pi$, denote the (unique) root of $\log[\cot(\rho/4)\cot(\theta/2)] - \theta/\sin \theta = 0$ lying in $(0, \pi - (\rho/2))$ by $\theta(\rho)$. On this interval the first term of the equation decreases from $+\infty$ to 0 while the left side increases from 1 to $(\pi - \rho/2)/\sin(\pi - (\rho/2))$. Given an arc $\Gamma \subseteq \partial \Delta$, with midpoint $\tau$, let $r_\tau$ denote the radius to $\tau$ and set $S(\alpha, \Gamma) = S(\alpha, \Gamma) \cup r_\tau$, $0 < \alpha < \pi.$
Theorem 1. Suppose \( f \in \mathfrak{D}(\rho) \) for some fixed \( \rho, 0 < \rho < 2\pi \), and some arc \( \Gamma_f \subset \partial \Delta \). Then there exists at least one point \( z_f \in \partial S(\theta(\rho), \Gamma_f) \) at which

\[
|f'(z_f)| (1 - |z_f|^2) > \frac{2}{e} \frac{\sin \theta(\rho)}{\theta(\rho)}. \tag{2.1}
\]

Proof. We may suppose \( \Gamma_f \) is symmetric about \( z = 1 \) so that \( r \) is the segment \( 0 < x < 1 \). To the contrary suppose for all \( z \in \partial S(\theta(\rho), \Gamma_f) \)

\[
|f'(z)| (1 - |z|^2) < \frac{2}{e} \frac{\sin \theta(\rho)}{\theta(\rho)}. \tag{2.2}
\]

Let \( x^* \) be the intersection of \( \partial S(\theta(\rho), \Gamma_f) \) and \([0, 1)\). We are going to show first that (2.2) and Theorem D-R imply \( |f(x^*)| > 1/e \); and secondly a straightforward estimate from (2.2) implies \( |f(x^*)| < 1/e \). To show the first inequality we put \( g = 1/f \) and let \( \Gamma_	heta \) denote the subarc of \( \Gamma_f \) with endpoints \( e^{-i\theta} \) and \( e^{i\theta} \), \( 0 < \theta < \rho/2 \). Put \( M(\theta) = \sup |g(z)|, z \in S(\theta(\rho), \Gamma_\theta) \). Choose any value \( \theta \) for which \( M(\theta) \) is finite. Apply Theorem D-R to \( g \) on the domain \( S(\theta(\rho), \Gamma_\theta) \). In this situation \( m = 1 \) and \( M = M(\theta) \) and so we conclude that for some \( q_\theta \in \partial S(\theta(\rho), \Gamma_\theta) \cap \Delta \)

\[
|g'(q_\theta)| (1 - |q_\theta|^2) > \frac{2 \sin \theta(\rho)}{\theta(\rho)} M(\theta) \log M(\theta). \tag{2.3}
\]

If we remember \( g = 1/f \) then (2.3) together with assumption (2.2) gives

\[
\frac{2 \sin \theta(\rho)}{\theta(\rho)} \frac{\log M(\theta)}{M(\theta)} \leq |f'(q_\theta)| (1 - |q_\theta|^2) < \frac{2}{e} \frac{\sin \theta(\rho)}{\theta(\rho)},
\]

or

\[
\frac{\log M(\theta)}{M(\theta)} \leq \frac{1}{e}. \tag{2.4}
\]

In the interval \([1, \infty)\) the function \( \log x/x \) has a single maximum value of \( 1/e \) at \( x = e \). The finite (and infinite) values of \( M(\theta) \) form a connected set, which because of (2.4) lies either in \([1, e)\) or \((e, \infty)\). But \( f \in \mathfrak{D}(\rho) \) implies that \( M(\theta) \) is close to \( 1 \) for small values of \( \theta \) and so we conclude that \( M(\theta) < e \) for all values of \( 0 < \theta < \rho/2 \). In particular \( 1/|g(x^*)| = |f(x^*)| > 1/e. \) In the other direction we estimate \( f(x^*) \) by integrating \( f'(x) \) along the interval \([0, x^*)\). Under assumption (2.2) and remembering that \( f(0) = 0 \) we have

\[
|f(x^*)| \leq \int_0^{x^*} |f'(x)| \, dx < \frac{2}{e} \frac{\sin \theta(\rho)}{\theta(\rho)} \int_0^{x^*} \frac{dx}{1 - x^2} = \frac{1}{e} \frac{\sin \theta(\rho)}{\theta(\rho)} \log \left( \frac{1 + x^*}{1 - x^*} \right). \tag{2.5}
\]

To solve for \( x^* \) in terms of \( \theta(\rho) \) and \( \rho \) recall that \( \partial S(\theta(\rho), \Gamma_f) \cap \Delta \) is part of a circle making an angle \( \theta(\rho) \) with \( \partial \Delta \) at \( e^{-i\rho/2} \) and \( e^{i\rho/2} \). Using the law of sines we calculate \( x^* \) as the difference between the distance of the center of the circle determined by \( \partial S(\theta(\rho), \Gamma_f) \) from the origin, and the radius of this circle. We obtain
that

\[ x^* = \frac{\sin(\theta(\rho)) - \sin(\rho/2)}{\sin(\theta(\rho) - (\rho/2))}. \]

A routine use of trigonometric identities shows that

\[ \frac{1 + x^*}{1 - x^*} = \cot\left(\frac{\theta(\rho)}{2}\right)\cot(\rho/4). \quad (2.6) \]

Because \( \theta(\rho) \) was chosen so that

\[ \sin \theta(\rho) = \cot(\rho/4) \]

(2.5) and (2.6) show that \( f(x^*) < 1/e \). Thus a contradiction is reached and (2.1) is established. Because \( 0 < \theta(\rho) < \pi - (\rho/2) \) the monotonicity of \( \sin x/x \) shows that (2.1) is a better estimate than (1.0). In fact it is a much stronger result especially for small values of \( \rho \). For example when \( \rho = \pi \) the constant in (1.0) is \( 4/e\pi \approx .468 \), while in (2.1) it is \( \approx .684 \); if \( \rho = \pi/10 \) then (1.0) gives a constant of \( \approx .08 \) while (2.1) gives \( \approx .345 \). As \( \rho \to 2\pi \), \( \theta(\rho) \to 0 \) and \( (2/e)\sin \theta(\rho)/\theta(\rho) \to 2/e \). In this asymptotic case the constant \( 2/e \) is best possible as was pointed out in [10]. Whether \( (2/e)\sin \theta(\rho)/\theta(\rho) \) is best possible in general we do not know. In Table 1 we give various values of \( \rho \), \( \theta(\rho) \) and \( \kappa(\rho) = (2/e)\sin \theta(\rho)/\theta(\rho) \). It is easy to generate extensive values of \( \kappa(\rho) \) with any computer. Numbers in the table have been rounded to three places.

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>( \theta(\rho) )</th>
<th>( \kappa(\rho) )</th>
</tr>
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<tr>
<td>( \pi )</td>
<td>.657</td>
<td>.684</td>
</tr>
<tr>
<td>( 4\pi/5 )</td>
<td>.839</td>
<td>.653</td>
</tr>
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</tr>
<tr>
<td>( 2\pi/5 )</td>
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<td>.538</td>
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<tr>
<td>( \pi/5 )</td>
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<td>.429</td>
</tr>
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<td>( \pi/50 )</td>
<td>2.35</td>
<td>.224</td>
</tr>
</tbody>
</table>

The convergence of \( \kappa(\rho) \) to 0 with \( \rho \) is rather slow. We have not been able to use functions belonging to the family \( \mathcal{D}(\rho) \) to obtain any improvement in the upper estimates for \( B \) and \( B_S \).

In Theorem 1 the normalization \( f(0) = 0 \) can be replaced by \( f(a) = 0 \) to produce a slightly more general theorem.

**Theorem 2.** Let \( f \) be holomorphic in \( \Delta \) and suppose \( f(a) = 0 \), \( a \in \Delta \). Suppose further for some arc \( \Gamma_f \subseteq \partial \Delta \) we have, for all \( t \in \Gamma_f \), \( \lim_{\zeta \to t} |f(\zeta)| > 1 \). Then there exists a point \( z_f \in S(\pi(1 - \omega(a, \Gamma_f, \Delta)), \Gamma_f) \) at which

\[ |f'(z_f)|(1 - |z_f|^2) > \kappa(2\pi\omega(a, \Gamma_f, \Delta)). \]

**Proof.** It is easy to see that \( g(\zeta) = f((\zeta + a)/(1 + \bar{a}\zeta)) \) is in \( \mathcal{D}(2\pi\omega(a, \Gamma_f, \Delta)) \) and of course \( |g'(\zeta)|(1 - |\zeta|^2) = |f'(z)|(1 - |z|^2), \ z = (\zeta + a)/(1 + \bar{a}\zeta) \). An application of Theorem 1 to \( g(\zeta) \) proves Theorem 2. (Actually \( z_f \) lies in a smaller
domain but for simplicity we use the more familiar albeit larger domain
\( S(\pi(1 - \omega(a, \Gamma_f, \Delta)), \Gamma_f). \)

We close with several questions on the classes \( \mathcal{P}(\rho) \). Can one say anything about
the boundary behavior of \( f \) away from the arc \( \Gamma_f \) on which \( |f| > 1 \)? By applying
Fatou's Theorem to \( 1/f \) in a neighborhood of \( \Gamma_f \) we see that \( f \) has angular limits
almost everywhere on \( \Gamma_f \). Can the condition \( \lim_{z \to e^i \Gamma_f} |f(z)| > 1 \) be relaxed to allow
this lower limit only for certain approaches to \( \Gamma_f \)? And lastly, is it possible to allow
\( \Gamma \) to be the union of finitely many arcs—perhaps symmetrically arranged on
\( \partial \Delta \)—and to produce a lower estimate for \( \|f\|_{B} \) which is better than the one obtained
by considering the largest subarc of \( \Gamma \)?

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