BANACH SPACES WITH THE 4.3. INTERSECTION PROPERTY

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Abstract. We show that a finite-dimensional Banach space has the 4.3. intersection property if and only if it is isometric to an $l_\infty$-sum of one- and two-dimensional spaces.

1. Introduction. Let $n$ and $k$ be integers with $n > k > 2$. We say that a Banach space $A$ has the $n,k$ intersection property (n.k.I.P.) if for every family $\{B(a_i, r_i)\}_{i=1}^n$ of $n$ closed balls in $A$ such that $\bigcap_{i=1}^k B(a_{i_1}, r_{i_1}) \neq \emptyset$ whenever $1 < i_1 < i_2 < \cdots < i_k < n$, we have $\bigcap_{i=1}^n B(a_i, r_i) \neq \emptyset$.

In this note we want to compute the structure of finite-dimensional real Banach spaces with the 4.3.I.P. We showed in [2] that complex spaces with the 4.3.I.P. are exactly the predual $L_1$-spaces. This is not true in the real case. By Helly's theorem [5] all one- and two-dimensional real spaces have the 4.3.I.P. Hence also $l_\infty$-sums of one- and two-dimensional spaces have the 4.3.I.P. In Theorem 3 we show a partial converse: If a finite-dimensional real space has the 4.3.I.P., then it is an $l_\infty$-sum of one- and two-dimensional spaces.

In the real case, a Banach space has the 4.2.I.P. if and only if it is a predual $L_1$-space [4]. But if $A \subseteq C(X)$, $X$ compact Hausdorff, and $I \in A$, then $A$ has the 4.3.I.P. if and only if it is a predual $L_1$-space [4].

In the following, let $A$ be a real Banach space. Closed balls in $A$ are denoted $B(a, r)$, and we write $A_1 = B(0, 1)$. The convex hull of a set $S$ is denoted conv$(S)$, and the set of extreme points of a convex set $C$ is denoted $\partial_e C$. Let $J$ be a subspace of $A$ and let $n > 2$. $H^n(A, J)$ denotes the space

$$H^n(A, J) = \left\{ (x_1, \ldots, x_n) : \text{all } x_i \in A \text{ and } \sum_{i=1}^n x_i \in J \right\}$$

equipped with the norm

$$\|(x_1, \ldots, x_n)\| = \sum_{i=1}^n \|x_i\|.$$ 

We also write $H^n(A) = H^n(A, \{0\})$.

We shall use the following result [1].

**Theorem 1.** If $A$ has the 4.3.I.P. and $(x_1, \ldots, x_n) \in \partial_e H^n(A^*)_1$, then at least one $x_i$ equals 0.
If in the definition of the 4.3.I.P. we replace \( \bigcap_{i=1}^{4} B(a_i, r_i) \neq \emptyset \) by

\[
\bigcap_{i=1}^{4} B(a_i, r_i + \varepsilon) \neq \emptyset
\]

for all \( \varepsilon > 0 \), then \( A \) has the 4.3.I.P. if and only if every extreme point in \( H^4(A^*)_1 \) has at least one component which is 0.

Since for each \( i, S_i = \{(x_1, \ldots, x_4) \in H^4(A^*)_1 : x_i = 0\} \) is a \( w^* \)-compact convex set, we get the following corollary.

**Corollary 2.** If \( A \) has the 4.3.I.P., then \( H^4(A^*)_1 = \text{conv}(\bigcup_{i=1}^{4} S_i) \).

2. **The main result.** Our main result is the following theorem.

**Theorem 3.** Assume \( A \) is a real finite-dimensional space. Then \( A \) has the 4.3.I.P. if and only if \( A \) is isometric to an \( l_\infty \)-sum of one- and two-dimensional spaces.

A subspace \( J \) of \( A \) is called an \( L \)-summand if there exists another subspace \( N \) such that \( A = J \oplus N \) and \( \|x + y\| = \|x\| + \|y\| \) for all \( x \in J \) and all \( y \in N \).

Clearly Theorem 3 follows from Proposition 4.

**Proposition 4.** Assume \( A \) has the 4.3.I.P. and let \( e \in \partial A^* \). Then there exists an \( L \)-summand \( J \) of \( A^* \) such that \( e \in J \) and \( \dim J = 1 \) or 2.

If \( J \) is a subspace of \( A \), let \( J' \) denote the (usually nonconvex) cone

\[
J' = \{ x \in A : x = 0 \text{ or } J \cap \text{face}(\|x\|^{-1}x) = \emptyset \}.
\]

For \( x \in A_1 \), \( \text{face}(x) \) denotes the smallest face of \( A_1 \) containing \( x \). A closed subspace \( J \) of \( A \) is called a semi \( L \)-summand if for all \( x \in J \) and all \( y \in J' \) we have \( \|x + y\| = \|x\| + \|y\| \). In [1] we proved that each \( L \)-summand \( J \) is a semi \( L \)-summand. \( N \) in the definition of \( L \)-summand is equal to \( J' \).

In the proof of Proposition 4 we will need the following result.

**Proposition 5.** Assume \( A \) has the 4.3.I.P. If \( J \) is a semi \( L \)-summand of \( A^* \), then \( J \) is an \( L \)-summand.

**Proof.** Let \( a \in J \) and let \( x, y \in J' \). By [1, Theorem 5.3] and [1, Lemma 5.4], it suffices to show that \( \|a + x + y\| = \|a\| + \|x + y\| \). By Corollary 2, there exist \((z_{1,1}, z_{1,2}, z_{1,3}, z_{1,4}) \in H^4(A^*) \) with \( z_{i,j} = 0 \) for \( i = 1, \ldots, 4 \), such that

\[
(a, x, y, -a - x - y) = (0, z_{1,2}, z_{1,3}, z_{1,4}) + (z_{2,1}, 0, z_{2,3}, z_{2,4}) + (z_{3,1}, z_{3,2}, 0, z_{3,4}) + (z_{4,1}, z_{4,2}, z_{4,3}, 0)
\]

and \( \|a\| = \|z_{2,1}\| + \|z_{3,1}\| + \|z_{4,1}\| \), \( \|x\| = \|z_{1,2}\| + \|z_{3,2}\| + \|z_{4,2}\| \) and so on. Since both \( J \) and \( J' \) are hereditary cones [1], we get \( z_{i,j} \in J \) and \( z_{i,2}, z_{i,3} \in J' \) for all \( i \). But then since \( J \) is a semi \( L \)-summand,

\[
\|z_{4,3}\| = \|z_{4,1} + z_{4,2}\| = \|z_{4,1}\| + \|z_{4,2}\|
\]

\[
= \|z_{4,1}\| + \|z_{4,1} + z_{4,3}\| = 2\|z_{4,1}\| + \|z_{4,3}\|.
\]
Hence $z_{4,1} = 0 = z_{4,2} + z_{4,3}$. Now using that $J$ is a semi $L$-summand several times gives

$$
\|a + x + y\| = \sum_{i=1}^{4} \|z_{i,4}\|
= \|z_{1,2} + z_{1,3}\| + \|z_{2,1} + z_{2,3}\| + \|z_{3,1} + z_{3,2}\|
= \|z_{1,2} + z_{1,3}\| + \|z_{2,1}\| + \|z_{2,3}\| + \|z_{3,1}\| + \|z_{3,2}\|
\geq \|z_{1,2} + z_{1,3} + z_{2,3} + z_{3,2}\| + \|z_{2,1} + z_{3,1}\|
= \|x + y\| + \|a\| \geq \|a + x + y\|.
$$

The proof is complete.

**Proof of Proposition 4.** Let $e \in \partial_{e}A_{1}^{*}$. If $\text{span}(e)$ is a semi $L$-summand, then by Proposition 5 there is nothing more to prove. Hence assume that $\text{span}(e)$ is not a semi $L$-summand. Then by [1, Corollary 5.13], there exists an $(x, y) \in \partial_{e}H^{2}(A^{*}, \text{span}(e))$, such that $x, y \not\in \text{span}(e)$ and $x + y \neq 0$. Let $z = -(x + y) \in \text{span}(e)$. Let $a(x,y,z) \in H^{1}(A^{*})$. By [3, Lemma 1], we get $a(x,y,z) \in \partial_{e}H^{2}(A^{*})$. Let $e_{1} = e$, $e_{2} = \|x\|^{-1}x$, $e_{3} = \|y\|^{-1}y$ and let $E = \text{span}(e_{1}, e_{2}, e_{3})$. Then $\dim E = 2$. By [2, Lemma 3.3], we also get $e_{2}, e_{3} \in \partial_{e}A_{1}^{*}$.

We claim that $E$ is an $L$-summand. By Proposition 5, it suffices to show that $E$ is a semi $L$-summand. Suppose $(u, v) \in \partial_{e}H^{2}(A^{*}, E)$. By [1, Corollary 5.13], $E$ is a semi $L$-summand if we can show that $u + v = 0$ or $u, v \in E$.

So suppose $u + v \neq 0$. Since $u + v \in E$, we can write $u + v = ae_{1} + be_{2}$. By using the basis $e_{1}, e_{2}, e_{3}$ for $E$ if necessary, we can assume $a \neq 0$ and $b \neq 0$. By Corollary 2, there exist $(z_{i,1}, z_{i,2}, z_{i,3}, z_{i,4}) \in H^{4}(A^{*})$ with $z_{i,i} = 0$ for $i = 1, \ldots, 4$ such that

$$
(u, v, -ae_{1}, -be_{2}) = (0, z_{1,2}, z_{1,3}, z_{1,4}) + (z_{2,1}, 0, z_{2,3}, z_{2,4})
+ (z_{3,1}, z_{3,2}, 0, z_{3,4}) + (z_{4,1}, z_{4,2}, z_{4,3}, 0)
$$

and $\|u\| = \|z_{2,1}\| + \|z_{3,1}\| + \|z_{4,1}\|, \|v\| = \|z_{2,1}\| + \|z_{3,2}\| + \|z_{4,2}\|$ and so on. Since $e_{1}, e_{2} \in \partial_{e}A_{1}^{*}$, we get $z_{i,3}, z_{i,4} \in E$ for all $i$. Hence

$$
(u, v) = (0, z_{1,2}) + (z_{2,1}, 0) + (z_{3,1}, z_{3,2}) + (z_{4,1}, z_{4,2})
$$
gives us a convex combination in $H^{2}(A^{*}, E)$. In fact,

$$
1 = \|u\| + \|v\| = \sum_{i=1}^{4} (\|z_{i,1}\| + \|z_{i,2}\|)
= \|(0, z_{1,2})\| + \|(z_{2,1}, 0)\| + \|(z_{3,1}, z_{3,2})\| + \|(z_{4,1}, z_{4,2})\|.
$$

If $(z_{3,1}, z_{3,2}) \neq 0$, then since $(u, v)$ is an extreme point, we have $(u, v) = t(z_{3,1}, z_{3,2})$ for some $t > 0$. Hence $ae_{1} + be_{2} = u + v = t(z_{3,1} + z_{3,2}) = -t(z_{3,4} = ce_{2}$ for some $c$. Since $e_{1}$ and $e_{2}$ are linearly independent, we get a contradiction. Hence $(z_{3,1}, z_{3,2}) = 0$. Similarly we show that $(z_{4,1}, z_{4,2}) = 0$. But then $u = z_{2,1} = -(z_{2,3} + z_{2,4}) \in E$ and $v = z_{1,2} \in E$. The proof is complete.

Let $A = (l_{\infty}^{2} \oplus R)_{1}^{*}$, $\dim A = 4$. Hence $A$ has the 6.5.I.P. An inspection of the extreme points of $H^{2}(A^{*})$, using a generalized version of [2, Lemma 3.3], shows that every extreme point of $H^{2}(A^{*})$ has at least one component which is 0. In fact,
if \((x_1, \ldots, x_d) \in \partial_x H^3(A^*), 1\) with all \(x_i \neq 0\), then \(\|x_i\|^{-1}x_i \in \partial_x A^*_1\) for all \(i\). (See [2, Lemma 3.3].) The extreme points of \(A^*_1\) are \((\pm 1, 0, 0 \pm 1), (0, \pm 1, 0 \pm 1)\) and \((0, 0 \pm 1, \pm 1)\). Since \(\Sigma_{j=1}^d x_j = 0\), we may assume that all \(x_i\) have a zero first coordinate. But then it follows that at least one \(x_i = 0\), since we can consider all \(x_i\) as vectors in \((l^2_1 \oplus R)_{\infty} = l^3_{\infty}\). A typical extreme point of \(H^3(A^*)\) is
\[((0, 0, 0, 0), (1, 0, 0, 1), (-1, 0, 0, 1), (0, 1, 0, -1), (0, -1, 0, -1)).\]

We get from [1, Theorem 2.10], that \(A\) has the 5.4.I.P. This example shows that if a finite-dimensional space \(A\) has the \((n + 1).n.I.P. with n > 4\), then it is not always possible to write \(A\) as an \(l^\infty\)-sum of spaces whose dimension is \(< n - 1\).

REFERENCES

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