# BANACH SPACES WITH THE 4.3. INTERSECTION PROPERTY 

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#### Abstract

We show that a finite-dimensional Banach space has the 4.3. intersection property if and only if it is isometric to an $l_{\infty}$-sum of one- and two-dimensional spaces.


1. Introduction. Let $n$ and $k$ be integers with $n>k \geqslant 2$. We say that a Banach space $A$ has the n.k. intersection property (n.k.I.P.) if for every family $\left\{B\left(a_{i}, r_{i}\right)\right\}_{i=1}^{n}$ of $n$ closed balls in $A$ such that $\cap_{r=1}^{k} B\left(a_{i}, r_{i}\right) \neq \varnothing$ whenever $1<i_{1}<i_{2}$ $\leqslant \cdots \leqslant i_{k} \leqslant n$, we have $\cap_{i=1}^{n} B\left(a_{i}, r_{i}\right) \neq \varnothing$.
In this note we want to compute the structure of finite-dimensional real Banach spaces with the 4.3.I.P. We showed in [2] that complex spaces with the 4.3.I.P. are exactly the predual $L_{1}$-spaces. This is not true in the real case. By Helly's theorem [5] all one- and two-dimensional real spaces have the 4.3.I.P. Hence also $l_{\infty}$-sums of one- and two-dimensional spaces have the 4.3.I.P. In Theorem 3 we show a partial converse: If a finite-dimensional real space has the 4.3.I.P., then it is an $l_{\infty}$-sum of one- and two-dimensional spaces.

In the real case, a Banach space has the 4.2.I.P. if and only if it is a predual $L_{1}$-space [4]. But if $A \subseteq C(X), X$ compact Hausdorff, and $l \in A$, then $A$ has the 4.3.I.P. if and only if it is a predual $L_{1}$-space [4].

In the following, let $A$ be a real Banach space. Closed balls in $A$ are denoted $B(a, r)$, and we write $A_{1}=B(0,1)$. The convex hull of a set $S$ is denoted $\operatorname{conv}(S)$, and the set of extreme points of a convex set $C$ is denoted $\partial_{e} C$. Let $J$ be a subspace of $A$ and let $n \geqslant 2 . H^{n}(A, J)$ denotes the space

$$
H^{n}(A, J)=\left\{\left(x_{1}, \ldots, x_{n}\right): \text { all } x_{i} \in A \text { and } \sum_{i=1}^{n} x_{i} \in J\right\}
$$

equipped with the norm

$$
\left\|\left(x_{1}, \ldots, x_{n}\right)\right\|=\sum_{i=1}^{n}\left\|x_{i}\right\| .
$$

We also write $H^{n}(A)=H^{n}(A,(0))$.
We shall use the following result [1].
Theorem 1. If $A$ has the 4.3.I.P. and $\left(x_{1}, \ldots, x_{4}\right) \in \partial_{e} H^{4}\left(A^{*}\right)_{1}$, then at least one $x_{i}$ equals 0.

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If in the definition of the 4.3.I.P. we replace $\cap_{i=1}^{4} B\left(a_{i}, r_{i}\right) \neq \varnothing$ by

$$
\bigcap_{i=1}^{4} B\left(a_{i}, r_{i}+\varepsilon\right) \neq \varnothing
$$

for all $\varepsilon>0$, then $A$ has the 4.3.I.P. if and only if every extreme point in $H^{4}\left(A^{*}\right)_{1}$ has at least one component which is 0 .

Since for each $i, S_{i}=\left\{\left(x_{1}, \ldots, x_{4}\right) \in H^{4}\left(A^{*}\right)_{1}: x_{i}=0\right\}$ is a $w^{*}$-compact convex set, we get the following corollary.

Corollary 2. If $A$ has the 4.3.I.P., then $H^{4}\left(A^{*}\right)_{1}=\operatorname{conv}\left(\cup_{i=1}^{4} S_{i}\right)$.
2. The main result. Our main result is the following theorem.

Theorem 3. Assume $A$ is a real finite-dimensional space. Then $A$ has the 4.3.I.P. if and only if $A$ is isometric to an $l_{\infty}$-sum of one- and two-dimensional spaces.

A subspace $J$ of $A$ is called an $L$-summand if there exists another subspace $N$ such that $A=J \oplus N$ and $\|x+y\|=\|x\|+\|y\|$ for all $x \in J$ and all $y \in N$.

Clearly Theorem 3 follows from Proposition 4.
Proposition 4. Assume $A$ has the 4.3.I.P. and let $e \in \partial_{e} A_{1}^{*}$. Then there exists an L-summand $J$ of $A^{*}$ such that $e \in J$ and $\operatorname{dim} J$ is 1 or 2.

If $J$ is a subspace of $A$, let $J^{\prime}$ denote the (usually nonconvex) cone

$$
J^{\prime}=\left\{x \in A: x=0 \text { or } J \cap \text { face }\left(\|x\|^{-1} x\right)=\varnothing\right\}
$$

For $x \in A_{1}$, face $(x)$ denotes the smallest face of $A_{1}$ containing $x$. A closed subspace $J$ of $A$ is called a semi $L$-summand if for all $x \in J$ and all $y \in J^{\prime}$ we have $\|x+y\|=\|x\|+\|y\|$. In [1] we proved that each $L$-summand $J$ is a semi $L$-summand. $N$ in the definition of $L$-summand is equal to $J^{\prime}$.

In the proof of Proposition 4 we will need the following result.
Proposition 5. Assume $A$ has the 4.3.I.P. If $J$ is a semi L-summand of $A^{*}$, then $J$ is an L-summand.

Proof. Let $a \in J$ and let $x, y \in J^{\prime}$. By [1, Theorem 5.3] and [1, Lemma 5.4], it suffices to show that $\|a+x+y\|=\|a\|+\|x+y\|$. By Corollary 2, there exist $\left(z_{i, 1}, z_{i, 2}, z_{i, 3}, z_{i, 4}\right) \in H^{4}\left(A^{*}\right)$ with $z_{i, i}=0$ for $i=1, \ldots, 4$ such that

$$
\begin{aligned}
(a, x, y,-a-x-y)= & \left(0, z_{1,2}, z_{1,3}, z_{1,4}\right)+\left(z_{2,1}, 0, z_{2,3}, z_{2,4}\right) \\
& +\left(z_{3,1}, z_{3,2}, 0, z_{3,4}\right)+\left(z_{4,1}, z_{4,2}, z_{4,3}, 0\right)
\end{aligned}
$$

and $\quad\|a\|=\left\|z_{2,1}\right\|+\left\|z_{3,1}\right\|+\left\|z_{4,1}\right\|, \quad\|x\|=\left\|z_{1,2}\right\|+\left\|z_{3,2}\right\|+\left\|z_{4,2}\right\|$ and so on. Since both $J$ and $J^{\prime}$ are hereditary cones [1], we get $z_{i, 1} \in J$ and $z_{i, 2}, z_{i, 3} \in J^{\prime}$ for all $i$. But then since $J$ is a semi $L$-summand,

$$
\begin{aligned}
\left\|z_{4,3}\right\| & =\left\|z_{4,1}+z_{4,2}\right\|=\left\|z_{4,1}\right\|+\left\|z_{4,2}\right\| \\
& =\left\|z_{4,1}\right\|+\left\|z_{4,1}+z_{4,3}\right\|=2\left\|z_{4,1}\right\|+\left\|z_{4,3}\right\| .
\end{aligned}
$$

Hence $z_{4,1}=0=z_{4,2}+z_{4,3}$. Now using that $J$ is a semi $L$-summand several times gives

$$
\begin{aligned}
\|a+x+y\| & =\sum_{i=1}^{4}\left\|z_{i, 4}\right\| \\
& =\left\|z_{1,2}+z_{1,3}\right\|+\left\|z_{2,1}+z_{2,3}\right\|+\left\|z_{3,1}+z_{3,2}\right\| \\
& =\left\|z_{1,2}+z_{1,3}\right\|+\left\|z_{2,1}\right\|+\left\|z_{2,3}\right\|+\left\|z_{3,1}\right\|+\left\|z_{3,2}\right\| \\
& \geqslant\left\|z_{1,2}+z_{1,3}+z_{2,3}+z_{3,2}\right\|+\left\|z_{2,1}+z_{3,1}\right\| \\
& =\|x+y\|+\|a\|>\|a+x+y\| .
\end{aligned}
$$

The proof is complete.
Proof of Proposition 4. Let $e \in \partial_{e} A_{1}^{*}$. If span(e) is a semi $L$-summand, then by Proposition 5 there is nothing more to prove. Hence assume that $\operatorname{span}(e)$ is not a semi $L$-summand. Then by [1, Corollary 5.13], there exists an $(x, y) \in$ $\partial_{e} H^{2}\left(A^{*}, \operatorname{span}(e)\right)_{1}$ such that $x, y \notin \operatorname{span}(e)$ and $x+y \neq 0$. Let $z=-(x+y) \in$ $\operatorname{span}(e)$. Let $\alpha^{-1}=\|x\|+\|y\|+\|z\|$. Then $\alpha(x, y, z) \in H^{3}\left(A^{*}\right)_{1}$. By [3, Lemma 1], we get $\alpha(x, y, z) \in \partial_{e} H^{3}\left(A^{*}\right)_{1}$. Let $e_{1}=e, e_{2}=\|x\|^{-1} x, e_{3}=\|y\|^{-1} y$ and let $E=\operatorname{span}\left(e_{1}, e_{2}, e_{3}\right)$. Then $\operatorname{dim} E=2$. By [2, Lemma 3.3], we also get $e_{2}, e_{3} \in \partial_{e} A_{1}^{*}$.

We claim that $E$ is an $L$-summand. By Proposition 5, it suffices to show that $E$ is a semi $L$-summand. Suppose $(u, v) \in \partial_{e} H^{2}\left(A^{*}, E\right)_{1}$. By [1, Corollary 5.13], $E$ is a semi $L$-summand if we can show that $u+v=0$ or $u, v \in E$.

So suppose $u+v \neq 0$. Since $u+v \in E$, we can write $u+v=a e_{1}+b e_{2}$. By using the basis $e_{1}, e_{3}$ or $e_{2}, e_{3}$ for $E$ if necessary, we can assume $a \neq 0$ and $b \neq 0$. By Corollary 2, there exist $\left(z_{i, 1}, z_{i, 2}, z_{i, 3}, z_{i, 4}\right) \in H^{4}\left(A^{*}\right)$ with $z_{i, i}=0$ for $i=$ $1, \ldots, 4$ such that

$$
\begin{aligned}
\left(u, v,-a e_{1},-b e_{2}\right)= & \left(0, z_{1,2}, z_{1,3}, z_{1,4}\right)+\left(z_{2,1}, 0, z_{2,3}, z_{2,4}\right) \\
& +\left(z_{3,1}, z_{3,2}, 0, z_{3,4}\right)+\left(z_{4,1}, z_{4,2}, z_{4,3}, 0\right)
\end{aligned}
$$

and $\|u\|=\left\|z_{2,1}\right\|+\left\|z_{3,1}\right\|+\left\|z_{4,1}\right\|,\|v\|=\left\|z_{1,2}\right\|+\left\|z_{3,2}\right\|+\left\|z_{4,2}\right\|$ and so on. Since $e_{1}, e_{2} \in \partial_{e} A_{1}^{*}$, we get $z_{i, 3}, z_{i, 4} \in E$ for all $i$. Hence

$$
(u, v)=\left(0, z_{1,2}\right)+\left(z_{2,1}, 0\right)+\left(z_{3,1}, z_{3,2}\right)+\left(z_{4,1}, z_{4,2}\right)
$$

gives us a convex combination in $H^{2}\left(A^{*}, E\right)_{1}$. In fact,

$$
\begin{aligned}
1 & =\|u\|+\|v\|=\sum_{i=1}^{4}\left(\left\|z_{i, 1}\right\|+\left\|z_{i, 2}\right\|\right) \\
& =\left\|\left(0, z_{1,2}\right)\right\|+\left\|\left(z_{2,1}, 0\right)\right\|+\left\|\left(z_{3,1}, z_{3,2}\right)\right\|+\left\|\left(z_{4,1}, z_{4,2}\right)\right\|
\end{aligned}
$$

If $\left(z_{3,1}, z_{3,2}\right) \neq 0$, then since $(u, v)$ is an extreme point, we have $(u, v)=t\left(z_{3,1}, z_{3,2}\right)$ for some $t>0$. Hence $a e_{1}+b e_{2}=u+v=t\left(z_{3,1}+z_{3,2}\right)=-t z_{3,4}=c e_{2}$ for some c. Since $e_{1}$ and $e_{2}$ are linearly independent, we get a contradiction. Hence ( $z_{3,1}, z_{3,2}$ ) $=0$. Similarly we show that $\left(z_{4,1}, z_{4,2}\right)=0$. But then $u=z_{2,1}=-\left(z_{2,3}+z_{2,4}\right) \in E$ and $v=z_{1,2} \in E$. The proof is complete.

Let $A=\left(l_{\infty}^{3} \oplus R\right)_{l}, \operatorname{dim} A=4$. Hence $A$ has the 6.5.I.P. An inspection of the extreme points of $H^{5}\left(A^{*}\right)_{1}$, using a generalized version of [2, Lemma 3.3], shows that every extreme point of $H^{5}\left(A^{*}\right)_{1}$ has at least one component which is 0 . In fact,
if $\left(x_{1}, \ldots, x_{5}\right) \in \partial_{e} H^{5}\left(A^{*}\right)_{1}$ with all $x_{i} \neq 0$, then $\left\|x_{i}\right\|^{-1} x_{i} \in \partial_{e} A_{1}^{*}$ for all $i$. (See [ 2 , Lemma 3.3].) The extreme points of $A_{1}^{*}$ are $( \pm 1,0,0 \pm 1),(0, \pm 1,0 \pm 1)$ and $(0,0 \pm 1, \pm 1)$. Since $\sum_{i=1}^{5} x_{i}=0$, we may assume that all $x_{i}$ have a zero first coordinate. But then it follows that at least one $x_{i}=0$, since we can consider all $x_{i}$ as vectors in $\left(l_{1}^{2} \oplus R\right)_{l_{\infty}}=l_{\infty}^{3}$. A typical extreme point of $H^{5}\left(A^{*}\right)_{5}$ is

$$
((0,0,0,0),(1,0,0,1),(-1,0,0,1),(0,1,0,-1),(0,-1,0,-1))
$$

We get from [1, Theorem 2.10], that $A$ has the 5.4.I.P. This example shows that if a finite-dimensional space $A$ has the $(n+1)$.n.I.P. with $n \geqslant 4$, then it is not always possible to write $A$ as an $l_{\infty}$-sum of spaces whose dimension is $\leqslant n-1$.

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