PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 3, November 1980

BANACH SPACES WITH THE 4.3. INTERSECTION PROPERTY

ÅSVALD LIMA

ABSTRACT. We show that a finite-dimensional Banach space has the 4.3. intersection property if and only if it is isometric to an l_∞-sum of one- and two-dimensional spaces.

1. Introduction. Let n and k be integers with $n > k \geqslant 2$. We say that a Banach space A has the n.k. intersection property (n.k.I.P.) if for every family $\{B(a_i, r_i)\}_{i=1}^n$ of n closed balls in A such that $\bigcap_{r=1}^k B(a_{i_r}, r_{i_r}) \neq \varnothing$ whenever $1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k \leqslant n$, we have $\bigcap_{i=1}^n B(a_i, r_i) \neq \varnothing$.

In this note we want to compute the structure of finite-dimensional real Banach spaces with the 4.3.I.P. We showed in [2] that complex spaces with the 4.3.I.P. are exactly the predual L_1-spaces. This is not true in the real case. By Helly's theorem [5] all one- and two-dimensional real spaces have the 4.3.I.P. Hence also l_∞-sums of one- and two-dimensional spaces have the 4.3.I.P. In Theorem 3 we show a partial converse: If a finite-dimensional real space has the 4.3.I.P., then it is an l_∞-sum of one- and two-dimensional spaces.

In the real case, a Banach space has the 4.2.I.P. if and only if it is a predual L_1-space [4]. But if $A \subseteq C(X)$, X compact Hausdorff, and $l \in A$, then A has the 4.3.I.P. if and only if it is a predual L_1-space [4].

In the following, let A be a real Banach space. Closed balls in A are denoted $B(a, r)$, and we write $A_1 = B(0, 1)$. The convex hull of a set S is denoted conv(S), and the set of extreme points of a convex set C is denoted $\partial_e C$. Let J be a subspace of A and let $n \geqslant 2$. $H^n(A, J)$ denotes the space

$$H^n(A, J) = \left\{(x_1, \ldots, x_n): \text{all } x_i \in A \text{ and } \sum_{i=1}^n x_i \in J\right\}$$

equipped with the norm

$$\|(x_1, \ldots, x_n)\| = \sum_{i=1}^n \|x_i\|.$$

We also write $H^n(A) = H^n(A, (0))$.

We shall use the following result [1].

THEOREM 1. *If A has the 4.3.I.P. and $(x_1, \ldots, x_4) \in \partial_e H^4(A^*)_1$, then at least one x_i equals* 0.

Received by the editors July 23, 1979 and, in revised form, October 12, 1979.
AMS (MOS) subject classifications (1970). Primary 46B99.
Key words and phrases. n. k. intersection property, L-summands.

If in the definition of the 4.3.I.P. we replace $\bigcap_{i=1}^4 B(a_i, r_i) \neq \varnothing$ by

$$\bigcap_{i=1}^{4} B(a_i, r_i + \varepsilon) \neq \varnothing$$

for all $\varepsilon > 0$, then A has the 4.3.I.P. if and only if every extreme point in $H^4(A^*)_1$ has at least one component which is 0.

Since for each i, $S_i = \{(x_1, \ldots, x_4) \in H^4(A^*)_1 : x_i = 0\}$ is a w^*-compact convex set, we get the following corollary.

COROLLARY 2. *If A has the* 4.3.*I.P., then* $H^4(A^*)_1 = \operatorname{conv}(\bigcup_{i=1}^4 S_i)$.

2. The main result. Our main result is the following theorem.

THEOREM 3. *Assume A is a real finite-dimensional space. Then A has the* 4.3.*I.P. if and only if A is isometric to an l_∞-sum of one- and two-dimensional spaces.*

A subspace J of A is called an L-summand if there exists another subspace N such that $A = J \oplus N$ and $\|x + y\| = \|x\| + \|y\|$ for all $x \in J$ and all $y \in N$.

Clearly Theorem 3 follows from Proposition 4.

PROPOSITION 4. *Assume A has the* 4.3.*I.P. and let $e \in \partial_e A_1^*$. Then there exists an L-summand J of A^* such that $e \in J$ and* $\dim J$ *is* 1 *or* 2.

If J is a subspace of A, let J' denote the (usually nonconvex) cone

$$J' = \left\{x \in A : x = 0 \text{ or } J \cap \operatorname{face}\left(\|x\|^{-1}x\right) = \varnothing\right\}.$$

For $x \in A_1$, face(x) denotes the smallest face of A_1 containing x. A closed subspace J of A is called a semi L-summand if for all $x \in J$ and all $y \in J'$ we have $\|x + y\| = \|x\| + \|y\|$. In **[1]** we proved that each L-summand J is a semi L-summand. N in the definition of L-summand is equal to J'.

In the proof of Proposition 4 we will need the following result.

PROPOSITION 5. *Assume A has the* 4.3.*I.P. If J is a semi L-summand of A^*, then J is an L-summand.*

PROOF. Let $a \in J$ and let $x, y \in J'$. By **[1**, Theorem 5.3] and **[1**, Lemma 5.4], it suffices to show that $\|a + x + y\| = \|a\| + \|x + y\|$. By Corollary 2, there exist $(z_{i,1}, z_{i,2}, z_{i,3}, z_{i,4}) \in H^4(A^*)$ with $z_{i,i} = 0$ for $i = 1, \ldots, 4$ such that

$$\begin{aligned}(a, x, y, -a - x - y) = (0, z_{1,2}, z_{1,3}, z_{1,4}) &+ (z_{2,1}, 0, z_{2,3}, z_{2,4}) \\ &+ (z_{3,1}, z_{3,2}, 0, z_{3,4}) + (z_{4,1}, z_{4,2}, z_{4,3}, 0)\end{aligned}$$

and $\|a\| = \|z_{2,1}\| + \|z_{3,1}\| + \|z_{4,1}\|$, $\|x\| = \|z_{1,2}\| + \|z_{3,2}\| + \|z_{4,2}\|$ and so on. Since both J and J' are hereditary cones **[1]**, we get $z_{i,1} \in J$ and $z_{i,2}, z_{i,3} \in J'$ for all i. But then since J is a semi L-summand,

$$\begin{aligned}\|z_{4,3}\| &= \|z_{4,1} + z_{4,2}\| = \|z_{4,1}\| + \|z_{4,2}\| \\ &= \|z_{4,1}\| + \|z_{4,1} + z_{4,3}\| = 2\|z_{4,1}\| + \|z_{4,3}\|.\end{aligned}$$

Hence $z_{4,1} = 0 = z_{4,2} + z_{4,3}$. Now using that J is a semi L-summand several times gives

$$\begin{aligned}\|a + x + y\| &= \sum_{i=1}^{4} \|z_{i,4}\| \\ &= \|z_{1,2} + z_{1,3}\| + \|z_{2,1} + z_{2,3}\| + \|z_{3,1} + z_{3,2}\| \\ &= \|z_{1,2} + z_{1,3}\| + \|z_{2,1}\| + \|z_{2,3}\| + \|z_{3,1}\| + \|z_{3,2}\| \\ &> \|z_{1,2} + z_{1,3} + z_{2,3} + z_{3,2}\| + \|z_{2,1} + z_{3,1}\| \\ &= \|x + y\| + \|a\| > \|a + x + y\|.\end{aligned}$$

The proof is complete.

PROOF OF PROPOSITION 4. Let $e \in \partial_e A_1^*$. If span(e) is a semi L-summand, then by Proposition 5 there is nothing more to prove. Hence assume that span(e) is not a semi L-summand. Then by [1, Corollary 5.13], there exists an $(x, y) \in \partial_e H^2(A^*, \text{span}(e))_1$ such that $x, y \notin \text{span}(e)$ and $x + y \neq 0$. Let $z = -(x + y) \in \text{span}(e)$. Let $\alpha^{-1} = \|x\| + \|y\| + \|z\|$. Then $\alpha(x, y, z) \in H^3(A^*)_1$. By [3, Lemma 1], we get $\alpha(x, y, z) \in \partial_e H^3(A^*)_1$. Let $e_1 = e$, $e_2 = \|x\|^{-1}x$, $e_3 = \|y\|^{-1}y$ and let $E = \text{span}(e_1, e_2, e_3)$. Then $\dim E = 2$. By [2, Lemma 3.3], we also get $e_2, e_3 \in \partial_e A_1^*$.

We claim that E is an L-summand. By Proposition 5, it suffices to show that E is a semi L-summand. Suppose $(u, v) \in \partial_e H^2(A^*, E)_1$. By [1, Corollary 5.13], E is a semi L-summand if we can show that $u + v = 0$ or $u, v \in E$.

So suppose $u + v \neq 0$. Since $u + v \in E$, we can write $u + v = ae_1 + be_2$. By using the basis e_1, e_3 or e_2, e_3 for E if necessary, we can assume $a \neq 0$ and $b \neq 0$. By Corollary 2, there exist $(z_{i,1}, z_{i,2}, z_{i,3}, z_{i,4}) \in H^4(A^*)$ with $z_{i,i} = 0$ for $i = 1, \ldots, 4$ such that

$$\begin{aligned}(u, v, -ae_1, -be_2) &= (0, z_{1,2}, z_{1,3}, z_{1,4}) + (z_{2,1}, 0, z_{2,3}, z_{2,4}) \\ &\quad + (z_{3,1}, z_{3,2}, 0, z_{3,4}) + (z_{4,1}, z_{4,2}, z_{4,3}, 0)\end{aligned}$$

and $\|u\| = \|z_{2,1}\| + \|z_{3,1}\| + \|z_{4,1}\|$, $\|v\| = \|z_{1,2}\| + \|z_{3,2}\| + \|z_{4,2}\|$ and so on. Since $e_1, e_2 \in \partial_e A_1^*$, we get $z_{i,3}, z_{i,4} \in E$ for all i. Hence

$$(u, v) = (0, z_{1,2}) + (z_{2,1}, 0) + (z_{3,1}, z_{3,2}) + (z_{4,1}, z_{4,2})$$

gives us a convex combination in $H^2(A^*, E)_1$. In fact,

$$\begin{aligned}1 = \|u\| + \|v\| &= \sum_{i=1}^{4} (\|z_{i,1}\| + \|z_{i,2}\|) \\ &= \|(0, z_{1,2})\| + \|(z_{2,1}, 0)\| + \|(z_{3,1}, z_{3,2})\| + \|(z_{4,1}, z_{4,2})\|.\end{aligned}$$

If $(z_{3,1}, z_{3,2}) \neq 0$, then since (u, v) is an extreme point, we have $(u, v) = t(z_{3,1}, z_{3,2})$ for some $t > 0$. Hence $ae_1 + be_2 = u + v = t(z_{3,1} + z_{3,2}) = -tz_{3,4} = ce_2$ for some c. Since e_1 and e_2 are linearly independent, we get a contradiction. Hence $(z_{3,1}, z_{3,2}) = 0$. Similarly we show that $(z_{4,1}, z_{4,2}) = 0$. But then $u = z_{2,1} = -(z_{2,3} + z_{2,4}) \in E$ and $v = z_{1,2} \in E$. The proof is complete.

Let $A = (l_\infty^3 \oplus R)_{l_1}$, $\dim A = 4$. Hence A has the 6.5.I.P. An inspection of the extreme points of $H^5(A^*)_1$, using a generalized version of [2, Lemma 3.3], shows that every extreme point of $H^5(A^*)_1$ has at least one component which is 0. In fact, if $(x_1, \ldots, x_5) \in \partial_e H^5(A^*)_1$ with all $x_i \neq 0$, then $\|x_i\|^{-1}x_i \in \partial_e A_1^*$ for all i. (See [2, Lemma 3.3].) The extreme points of A_1^* are $(\pm 1, 0, 0 \pm 1)$, $(0, \pm 1, 0 \pm 1)$ and $(0, 0 \pm 1, \pm 1)$. Since $\Sigma_{i=1}^5 x_i = 0$, we may assume that all x_i have a zero first coordinate. But then it follows that at least one $x_i = 0$, since we can consider all x_i as vectors in $(l_1^2 \oplus R)_{l_\infty} = l_\infty^3$. A typical extreme point of $H^5(A^*)_5$ is

$$((0, 0, 0, 0), (1, 0, 0, 1), (-1, 0, 0, 1), (0, 1, 0, -1), (0, -1, 0, -1)).$$

We get from [1, Theorem 2.10], that A has the 5.4.I.P. This example shows that if a finite-dimensional space A has the $(n + 1)$.n.I.P. with $n > 4$, then it is not always possible to write A as an l_∞-sum of spaces whose dimension is $< n - 1$.

References

1. Å. Lima, *Intersection properties of balls and subspaces in Banach spaces*, Trans. Amer. Math. Soc. **227** (1977), 1–62.

2. ______, *Complex Banach spaces whose duals are L_1-spaces*, Israel J. Math. **24** (1976), 59–72.

3. ______, *An application of a theorem of Hirsberg and Lazar*, Math. Scand. **38** (1976), 325–340.

4. J. Lindenstrauss, *Extensions of compact operators*, Mem. Amer. Math. Soc. No. 48 (1964).

5. E. Helly, *Über Mengen konvexer Körper mit gemeinschaftlichen Punkten*, Jber. Deutsch. Math.-Verein. **32** (1923), 175–176.

Department of Mathematics and Statistics, Agricultural University of Norway, 1432 AAS-NLH, Norway