A LOWER BOUND FOR THE SPECTRAL RADIUS

VLASTIMIL PTÁK

ABSTRACT. We prove an inequality for a problem of Carathéodory type: given \( n \) inner functions \( m_1, m_2, \ldots, m_n \), to find the smallest norm of an \( H^\infty \) function such that the first \( n \) terms of its power series coincide with those of the product \( m_1 \cdots m_n \). As an application, we obtain a lower bound for the spectral radius of an \( n \)-dimensional operator on Hilbert space in terms of its norm and the norm of its \( n \)th power.

In 1960 J. Mařík and the present author proved [4] that a linear operator \( A \) on \( n \)-dimensional \( l^\infty \) space which satisfies \( |A| = |A|^q = 1 \) for some \( q > n^2 - n + 1 \) has an eigenvalue of modulus one. Later [6] the present author introduced, for each Banach space \( E \), the critical exponent of \( E \) as the smallest natural number \( q \) for which the following implication holds. If \( A \) is a linear operator on \( E \) and \( |A| = |A|^q = 1 \) then \( |A|^q = 1 \). (We denote by \( |A| \) the norm of \( A \) as a linear operator on \( E \) and by \( |A|_0 \) its spectral radius.) In the same paper [6] the author proved that the critical exponent of Hilbert space equals its dimension. This result says the following. Consider a linear operator \( A \) in \( n \)-dimensional Hilbert space with \( |A| = 1 \); if \( |A|^q \) is still one then \( A^n \) does not tend to zero as \( m \to \infty \). In other words, if \( |A| = 1 \) and if the powers converge to zero then they start converging with the \( n \)th term at the latest. This, of course, is only a qualitative result. What would be of more use, however, is a quantitative version of it. If the convergence is still bad at the critical exponent \( q \) (i.e. if \( |A|^q \) is still large) then it will continue being bad since we can infer from the behaviour of the first \( q \) terms that the spectral radius is close to one. To measure exactly how close to one, the author raised, in the next paper of the series [8], the following problem. Given a Banach space \( E \), \( 0 < p < 1 \), and a natural number \( r \) compute \( \sup \{|A|^r|\} \) where \( A \) ranges over all linear operators on \( E \) with \( |A| = 1 \) and \( |A|_0 < p \). A simple compactness argument shows that this supremum \( C(E, p, z^r) \) is \( < 1 \) for each \( p < 1 \) provided \( r \) is the critical exponent of \( E \) or some greater number.

In general, we can define, for each Banach space \( E \), \( 0 < p < 1 \), and each \( f \) holomorphic in a neighbourhood of the disc \( |z| < p \),

\[
C(E, p, f) = \sup \{|f(A)|; |A| < 1, |A|_0 < p\}.
\]

In the same paper [8] this supremum was computed for the particular case of \( n \)-dimensional Hilbert space and the \( n \)th power—computed in the sense that an
operator \( W \) was constructed for which \(|W^n|\) is a maximum among all contractions on \( n \)-dimensional Hilbert space whose spectral radius does not exceed \( p \).

With respect to a suitable basis [10] the matrix of \( W \) assumes a fairly simple form so that the \( n \)th power may be computed. The result is as follows. Consider a positive \( p < 1 \) and a natural number \( n \). Let \( m \) be the function (holomorphic in a neighbourhood of the closed unit disc) defined as

\[
m(z) = \frac{(z + p)}{(1 + pz)}.
\]

Denote by \( c_0, c_1, c_2, \ldots \) the coefficients of the Taylor expansion \( m(z)^n = c_0 + c_1z + c_2z^2 + \cdots \). If we write, for shortness, \( C_n(p) = C(H_n, p, z^n) \) then \( C_n(p) = |T| \) where \( T \) is the Toeplitz matrix

\[
\begin{bmatrix}
c_0 & 0 & 0 & \cdots & 0 \\
c_1 & c_0 & 0 & \cdots & 0 \\
c_2 & c_1 & c_0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c_{n-1} & c_{n-2} & c_{n-3} & \cdots & c_0
\end{bmatrix}
\]

In the present paper these results will be used to determine the asymptotic behaviour of \( C(H_n, p, z^n) \) as \( p \) tends to zero. The main theorem is based on a simple inequality for inner functions which seems to be of independent interest. This inequality is related to the classical Carathéodory problem and may be stated as follows. Given \( n \) inner functions \( m_1, \ldots, m_n \) then there exists a function holomorphic in a neighbourhood of the unit disc such that the first \( n \) terms of its development coincide with those of the product \( m_1 \cdots m_n \) and such that its maximum on the closed unit disc does not exceed \( \sum_{k=1}^{n} |m_k(0)| \).

**Notation.** If \( E \) is a Banach space we denote by \( \mathcal{L}(E) \) the algebra of all bounded linear operators on \( E \) with the operator norm. We denote by \( H_n \) the \( n \)-dimensional Hilbert space, by \( C^n \) the concrete Hilbert space of \( n \)-dimensional column vectors with the \( l^2 \) norm. The unit vectors in \( C^n \) will be denoted by \( e_1, \ldots, e_n \). By \( H^2 \) and \( H^{\infty} \) we denote the Hardy spaces, the corresponding norms will be written as \(|\cdot|_2\) and \(|\cdot|_\infty\). If \( f \in H^{\infty} \) we define \( Sf \) to be the function

\[
S(f) = \frac{(f(z) - f(0))}{z}.
\]

The operator \( S \) is a contraction in \( H^2 \).

1. **Products of inner functions.** This section is devoted to the proof of an inequality for products of inner functions which forms the basis of the estimate of \( C_n(p) \). The inequality seems to be interesting in its own right.

If \( f \in H^{\infty} \) and \( n \) is a given natural number we shall denote by \( \beta_n(f) \) the norm of \( f \) in the quotient space \( H^{\infty} / z^n H^{\infty} \),

\[
\beta_n(f) = \inf_{k \in H^{\infty}} |f - z^k|_\infty.
\]

Let us remark here that it can be shown that the infimum is attained.
(1.1) Proposition. Let $m_1, \ldots, m_n$ be given inner functions. Then there exists a $k \in H^\infty$ such that

$$|m_1 \cdots m_n - z^nk|_\infty < \sum_{i=1}^{n} |m_i(0)|.$$  

Proof. By induction. Since $m_1 - zS(m_1) = m_1(0)$ we can set $k_1 = S(m_1)$. Suppose that $n > 1$ and that $k_n \in H^\infty$ is such that the function $f_n = m_1 \cdots m_n - z^n k_n$ has $H^\infty$ norm $|f_n|_\infty < \Sigma_{i=1}^{n} |m_i(0)|$. Then

$$m_1 \cdots m_n m_{n+1} - z^{n+1} S(m_{n+1}k_n)$$

$$= (f_n + z^n k_n) m_{n+1} - z^n (m_{n+1}k_n - m_{n+1}(0)k_n(0))$$

$$= f_n m_{n+1} + z^n m_{n+1}(0)k_n(0)$$

so that, setting $k_{n+1} = S(m_{n+1}k_n)$, the following estimate holds:

$$|m_1 \cdots m_{n+1} - z^{n+1} k_{n+1}|_\infty < |f_n|_\infty + |m_{n+1}(0)||k_n(0)|.$$  

It follows that the induction will be complete if we prove that $|k_n(0)| < 1$. Since $|k_n(0)| < |k_n|_2$ this will be a consequence of the following stronger assertion: $|k_n|_2 < 1$ for all $n$. Now $|k_n|_2 = |S(m_1)|_2 < |m_1|_2 = 1$ and $|k_{n+1}|_2 = |S(m_{n+1}k_n)|_2 < |m_{n+1}k_n|_2 < |k_n|_2$ so that the proof is complete.

The preceding proposition may be reformulated as follows.

Given $n$ inner functions $m_1, \ldots, m_n$ then

$$\beta_n(m_1 \cdots m_n) < \sum_{i=1}^{n} |m_i(0)|.$$  

A closer look at the proof reveals the fact that equality is never attained here except in very special cases.

First of all, if $n = 1$, we have $\beta_1(m) = |m(0)|$ for every inner function $m$. Indeed, given $k \in H^\infty$, we have $|m(0)| = |(m - zk)(0)| < |m - zk|_\infty$ so that $|m(0)| < \beta_1(m)$. At the same time, for $k = S(m)$, we have $m - zk = m(0)$ whence $\beta_1(m) < |m(0)|$.

Thus for one function, we always have equality. The following result shows that, for $n > 1$, we have strict inequality almost always.

(1.2) Proposition. If $n > 1$ then

$$\beta_n(m_1 \cdots m_n) = \sum_{i=1}^{n} |m_i(0)|$$

if and only if one of the following situations is obtained.

1°. All $m_j(0)$ are zero.

2°. Exactly one $m_j(0)$ is different from zero all other $m_q$ being of the form $e_q z$ with $|e_q| = 1$.

Proof. If all $m_j(0)$ are zero there exist inner functions $a_j$ such that $m_j(z) = za_j(z)$. Setting $k = a_1 \cdots a_n$ we have $m_1 \cdots m_n - z^nk = 0$ so that $\beta_n(m_1, \ldots, m_n)$ is zero and so is $\Sigma |m_j(0)|$.  

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Suppose now that \( m_j(z) = z^j h_j(z) \) with \( e_j > 1 \) for \( j = 1, 2, \ldots, n - 1 \) and that \( m_n(0) \neq 0 \). Set \( w = e_1 + \cdots + e_{n-1} \) and \( h = h_1 \cdots h_{n-1} \) so that \( w > n - 1 \). We then have
\[
m_1 \cdots m_n - z^n k = z^n h m_n - z^n k = z^{n-1} (z^{w-n} + 1) h m_n - zk.
\]

Let us distinguish the following cases:

If \( w = n - 1 \) then \( |m_1 \cdots m_n - z^n k|_\infty = |h m_n - zk|_\infty \) so that \( \beta_n(m_1 \cdots m_n) = \beta_1(h m_n) = |h(0)m_n(0)| \).

If \( h \) is constant then
\[
\beta_n(m_1 \cdots m_n) = |m_n(0)| = \sum |m_j(0)|.
\]

If \( h \) is nonconstant we have \( |h(0)| < 1 \) so that
\[
\beta_n(m_1 \cdots m_n) < |m_n(0)| = \sum |m_j(0)|.
\]

If \( w > n \) we have \( m_1 \cdots m_n - z^n k = z^n(z^{w-n} h m_n - k) \) so that
\[
\beta_n(m_1 \cdots m_n) = 0 < \sum |m_j(0)|.
\]

We see thus that, in the case that exactly one factor is not divisible by \( z \), we always have sharp inequality except when all divisible \( m_j \) are of the form \( e_j z \) with \( |e_j| = 1 \).

The proof will be complete if we show that we have sharp inequality if at least two \( m_j(0) \) are different from zero. Hence suppose that \( m_1(0) \neq 0 \) and \( m_2(0) \neq 0 \). If \( m_1 \) is constant we have, for \( k = 0 \),
\[
\beta_2(m_1 m_2) < |m_1 m_2|_\infty = 1 < 1 + |m_2(0)| = |m_1(0)| + |m_2(0)|.
\]

If both \( m_1 \) and \( m_2 \) are nonconstant, we have, setting \( k = S(S(m_1)m_2) \),
\[
m_1(z)m_2(z) - z^2 k(z) = m_1(0)m_2(z) + zm_1'(0)m_2(0)
\]
whence \( \beta_2(m_1, m_2) < |m_1(0)| + |m_1'(0)| |m_2(0)| \).

Since \( |m_1|_2 < 1 \) and \( m_1(0) \neq 0 \) we have \( |m_1'(0)| < 1 \). It follows from this and from \( m_2(0) \neq 0 \) that \( \beta_2(m_1, m_2) < |m_1(0)| + |m_2(0)| \).

The rest follows from an inequality which we have already used.

In fact, since in each case we have found \( k \in H^\infty \) with \( |m_1 m_2 - z^2 k|_\infty < |m_1(0)| + |m_2(0)| \) and \( |k|_2 < 1 \), we may continue the construction used to prove Proposition (1.1) and conclude that
\[
\beta_n(m_1 \cdots m_n) < |m_1(0)| + |m_2(0)| + \cdots + |m_n(0)|.
\]

2. Asymptotics of \( C_n(p) \). We shall assume \( n > 1 \) since the case \( n = 1 \) presents little interest.

The asymptotic behaviour of \( C_n(p) \) as \( p \) tends to zero is completely described by the following

(2.1) Proposition. For \( 0 < p < 1 \),
\[
n p > C_n(p) > np(1 - p^2 g(p))
\]
where \( \lim g(p) = n(n - 1)/2 \) as \( p \) tends to zero.
Proof. The lower estimate is a consequence of the characterization of the extremal operator for $C_n(p)$. Indeed, we have

$$C_n(p) = |T_n(p)| > (T_n(p)e_n, e_1) = c_{n-1}$$

$$= \sum_{i=1}^{n-1} (-1)^{i-1} \binom{n}{t} \binom{n-2}{t-1} p^{2t-1}(1 - p^2)^{n-t}$$

which yields the desired inequality for small $p$. The formula for $c_{n-1}$ is obtained after a little calculation with the coefficients of $(z + p)/(1 + pz)$.

Let us prove now, for all $0 < p < 1$, the inequality $C_n(p) < np$. To this end, consider the shift matrix

$$E = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and observe that $T_n(p) = m^n(E)$. By Proposition (1.1) there exists a $k_n \in H^\infty$ such that the function $f_n = m^n - z^n k_n$ has norm $|f_n|_\infty < n|m(0)| = np$. Since $E^n = 0$ we have $m^n(E) = f_n(E)$. By the von Neumann inequality $C_n(p) = |T_n(p)| = |m^n(E)| = |f_n(E)| < |f_n|_\infty < np$. This completes the proof.

(2.2) Corollary. Let $A$ be a nonzero linear operator on $n$-dimensional Hilbert space. Then

$$n^{-1}|A|^{1-\alpha}|A^n| < |A|_\alpha.$$

The coefficient $n^{-1}$ is the best possible: If $\alpha$ satisfies $\alpha|A|^{1-\alpha}|A^n| < |A|_\alpha$ for all nonzero $A \in B(H_n)$ then $\alpha < n^{-1}$.

The inequality is strict for $n > 1$.

Proof. Set $p = |A|^{-1}|A|_\alpha$ so that $0 < p < 1$. The cases $p = 0$ and $p = 1$ being easy, let us assume $0 < p < 1$. Set $W = |A|^{-1}A$ so that $|W| = 1$ and $|W|_\alpha = p$. Then

$$|A|^{1-\alpha}|A^n| = |A| |W^n| < |A|C_n(p) < |A|n|A|^{-1}|A|_\alpha = n|A|_\alpha$$

whence the assertion. That no greater constant than $n^{-1}$ will do is a consequence of the other inequality in Proposition (2.1).

Added in Proof. Following a lecture given by the author about this matter in Bucharest, C. Apostol found an inequality for the trace norm of an operator from which $C_n(p) < np$ can also be deduced. This inequality is reproduced in the survey paper V. Pták and N. J. Young, Functions of operators and the spectral radius accepted by Linear Algebra and its Applications.

References

1. Z. Dostál, Uniqueness of the operator attaining $C(H_n, p, z^n)$, Časopis Pěst. Mat. 103 (1978), 236–243.


**Mathematics Institute Academy, Žitná 25, 115 67 Praha 1, Czechoslovakia**