ON ORDERED HARMONIC BOUNDED VARIATION

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Abstract. An example is given of a continuous real function that is of ordered harmonic bounded variation but not of harmonic bounded variation.

Let \( f \) be a real-valued function defined on \([0, 1]\), and for each open interval \( I = (a, b) \subset [0, 1] \) set \( f(I) = f(b) - f(a) \). Then \( f \) is said to be of harmonic bounded variation (HBV) on \([0, 1]\) if

\[
\sup \sum |f(I_n)|/n < \infty, \tag{\dagger}
\]

where the supremum is taken over all sequences of nonoverlapping open intervals \( I_1, I_2, \ldots \) in \([0, 1]\); if \((\dagger)\) holds when the supremum is taken over all sequences of open intervals \( I_1, I_2, \ldots \) in \([0, 1]\) for which either \( I_j < I_{j+1} \) for each index \( j \) or \( I_{j+1} < I_j \) for each index \( j \) (where \( I < J \) means \( I \) lies to the left of \( J \)), then \( f \) is said to be of ordered harmonic bounded variation (OHBV) on \([0, 1]\). These two function classes were introduced by D. Waterman in [1] and [2], respectively, and in [2] he asked whether the inclusion \( HBV \subset OHBV \) is proper. Here we show that it is.

Lemma. Let \( N \) be a nonnegative integer, \( M \) a positive integer, and \( 0 < \epsilon < 1 \). Then there exists an integer \( T > M \) and a corresponding sequence of positive numbers \( A_{N+1} > A_{N+2} > \cdots > A_{N+T} \) for which the following hold:

(i) \( \sum_{j=1}^{M} A_{N+j}/j < \epsilon \),
(ii) \( \sum_{j=1}^{T} A_{N+j}/(N+j) > 1 \) but \( \sum_{j=1}^{T} A_{N+j}/j < 4 \),
(iii) \( \sum_{j=1}^{T} A_{N+i+1-i}/j < 6\epsilon \) \( (1 < i < T) \).

Proof. Define \( A_{N+1} = A_{N+2} = \cdots = A_{N+M} = \epsilon/(N+M+1) \) and

\[
A_j = A_{N+j}/k \quad \text{for} \quad (N + M)2^{k-1} < j < (N + M)2^k \quad (k > 1).
\]

Noting that (i) is satisfied, we proceed to prove (ii).

First observe that if \( N + T = (N + M)2^K \quad (K > 1) \), then

\[
\sum_{j=1}^{T} A_{N+j}/(N+j) = A_{N+1} \left\{ \sum_{j=1}^{M} 1/(N+j) + \sum_{k=1}^{K} \left( \frac{1}{k} \right) \sum_{j=(N+M)2^{k-1}}^{(N+M)2^k} \right\}, \tag{*}
\]

where we use the notation \( \sum(a, b) = \sum_{j=a+1}^{b} 1/j \). For \( K = 1 \), the right-hand side of \((*)\) is \( < A_{N+1}(M+1) \), which by the definition of \( A_{N+1} \) is \( < 1 \). Also, since

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In 2 > \sum (n, 2n) \to \ln 2 \text{ as } n \to \infty,

the second series on the right-hand side of (**) is a partial sum of a divergent series
with its kth term (k > 1) less than \((\ln 2)/2\); hence, for some \(T\) we have

\[ 1 < \sum_{j=1}^{T} A_{N+j}/(N+j) < 2, \]  
(***)

which proves the first part of (ii). The second part follows from (***), the simple
observation that

\[ \sum_{j=1}^{T} A_{N+j}/j - \sum_{j=1}^{T} A_{N+j}/(N+j) = \sum_{j=1}^{T} NA_{N+j}/j(N+j) < \sum_{j=1}^{T} \epsilon/j(N+j) < 2, \]

To begin consideration of (iii) we note that \(A_{N+t+1-j} = A_{N+j}\) for \(j < t < M,\)
and hence, that (iii) for \(1 < t < M\) follows from (i).

Now suppose \(t\) satisfies \((N + M)2^{K-1} < N + t < (N + M)2^K\) for some \(K > 1.\)
Then let \(S_n\) \((n = 1, 2, \ldots, K)\) denote the sum

\[ \sum_{j} A_{N+t+1-j}/j \text{ for } (N + M)2^{K-n} < N + t + 1 - j < (N + M)2^{K-n+1}, \]

and let \(S_{K+1}\) denote this sum for \(N + 1 < N + t + 1 - j < N + M.\)

Consider \(S_1.\) For the indices \(j\) involved in this sum, we have \(A_{N+t+1-j} = A_{N+1}/K\) and hence

\[ S_1 \leq (A_{N+1}/K)
\leq [(\epsilon/K(N + M + 1)] \leq 2e.\]

Similarly \(S_2 < 2e.\)

Now consider \(S_n\) for \(3 < n < K.\) Since \(A_{N+t+1-j} = A_{N+1}/(K - n + 1)\) for each
of the \((N + M)2^{K-n}\) indices \(j\) involved in \(S_n\) and since the first of these indices is
greater than \((N + M)2^{K-2},\) which is the number of terms in \(S_2,\) we see that

\[ S_n \leq [A_{N+1}/(K - n + 1)] [(N + M)2^{K-n}/(N + M)2^{K-2}] \]

\[ = [A_{N+1}/(K - n + 1)]/2^{n-2}. \]

Then, since \(A_{N+1} < \epsilon,\) it follows that \(S_n < \epsilon/2^{n-2}\) for \(3 < n < K.\)

Combining these estimates with the observation that \(S_{K+1}\) is dominated by
\(\sum_{j=1}^{K} A_{N+j}/j\) which by (i) is \(< \epsilon,\) we have

\[ \sum_{j=1}^{T} A_{N+t+1-j}/j < 2\epsilon + 2\epsilon + \epsilon \sum_{n=3}^{K} 2^{2-n} + \epsilon < 6\epsilon, \]

and the lemma is proved.

**Theorem.** There exists a continuous \(f \in OHBV - HBV.\)

**Proof.** Choose positive numbers \(\epsilon_1, \epsilon_2, \ldots\) so that \(\sum_{n=1}^{\infty} \epsilon_n = 1.\) Then apply the
Lemma for \(N = 0, M = M_1 \equiv 1, \epsilon = \epsilon_1\) to obtain an integer \(T_1 > 1\) and a
sequence of positive numbers $A_1 > A_2 > \cdots > A_{T_1}$ such that
(i) $\sum_{j=1}^{M_2} A_j/j < \varepsilon_1$,
(ii) $1 < \sum_{j=1}^{T_2} A_j/j < 4$,
(iii) $\sum_{j=1}^{T_2} A_{T_1+j}/j < 6\varepsilon_1 (1 < t < T_1)$.
Now choose $M_2 > T_1$ so large that
$$\sum_{j=1}^{T_1} A_j/(M_2 + j) < \varepsilon_1.$$ Then apply the Lemma for $N = T_1, M = M_2, \varepsilon = \varepsilon_2$ to obtain an integer $T_2 > M_2$ and a sequence of positive numbers $A_{T_1+1} > A_{T_1+2} > \cdots > A_{T_1+T_2}$ such that
(i) $\sum_{j=1}^{M_2} A_{T_1+j}/j < \varepsilon_2$,
(ii) $1 < \sum_{j=1}^{T_2} A_{T_1+j}/(T_1 + j)$ but $\sum_{j=1}^{T_2} A_{T_1+j}/j < 4$,
(iii) $\sum_{j=1}^{T_2} A_{T_1+j+1}/j < 6\varepsilon_2 (1 < t < T_2)$.
Now choose $M_3 > T_2$ so large that
$$\sum_{j=1}^{T_2} A_{T_1+j}/(M_3 + j) < \varepsilon_2.$$ Then apply the Lemma for $N = T_1 + T_2, M = M_3, \varepsilon = \varepsilon_3$ and continue inductively to obtain: two sequences of positive integers $M_1, M_2, \ldots$ and $T_1, T_2, \ldots$ satisfying
$$M_k < T_k < M_{k+1} \quad (k > 1),$$ and a sequence of positive numbers $A_1, A_2, \ldots$ satisfying
$$A_{a_k+1} > A_{a_k+2} > \cdots > A_{a_k+T_k} \left( \sigma_k \equiv \sum_{j=0}^{k-1} T_j; T_0 \equiv 0; k > 1 \right)$$ such that
(a) $\sum_{j=1}^{M_k} A_{a_k+j}/j < \varepsilon_k$,
(b) $\sum_{j=1}^{T_k} A_{a_k+j}/(\sigma_k + j) > 1$ but $\sum_{j=1}^{T_k} A_{a_k+j}/j < 4$,
(c) $\sum_{j=1}^{T_k} A_{a_k+j}/(M_{k+1} + j) < \varepsilon_k$,
(d) $\sum_{j=1}^{M_k} A_{a_k+i+j+1}/j < 6\varepsilon_k (1 < i < T_k)$.

Now we proceed to define the desired function. Inductively we obtain a sequence of mutually disjoint closed intervals $I_n \equiv [a_n, b_n]$ contained in $(0, 1)$ such that for each index $k$ we have
$$I_i < I_{a_k+i} < I_{a_k+i+1} \quad (j = 1, 2, \ldots, T_k - 1; i > \sigma_{k+1}).$$ Then we define $f(x) = 0$ if $x \notin (a_n, b_n)$ for each $n$, $f((a_n + b_n)/2) = A_n$, and linearly extend $f$ to the remainder of $[0, 1]$.

For each index $n$ set $I_n^* = (a_n, (a_n + b_n)/2)$ and note that, by the first inequality in (b), we have
$$\sum_{n=1}^{\infty} f(I_n^*)/n = \sum_{n=1}^{\infty} A_n/n = \sum_{k=1}^{\infty} \sum_{j=1}^{T_k} A_{a_k+j}/(\sigma_k + j) = \infty;$$ that is, $f \notin HBV$.  

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To show that $f \in OHBV$, it suffices to prove the finiteness of the supremum of the sums $\sum f(J_n)/n$ taken over all ordered sequences of intervals $J_1, J_2, \ldots$ from the collection $(I_*)_{n=1}^{\infty}$.

First suppose $J_1 > J_2 > \cdots$. Then it follows readily from (d) that

$$\sum_{k=1}^{\infty} f(J_n)/n < 6 \sum_{k=1}^{\infty} e_k = 6.$$ 

Now suppose $J_1 < J_2 < \cdots$. Choose an integer $k_0$ and let $J_{n_0+1}, J_{n_0+2}, \ldots, J_{n_0+t_0}$ be the $J_n$'s that equal one of the intervals $I_{*n_0+j} (j = 1, 2, \ldots, T_{k_0})$. Since $A_{*n_0+j} > A_{*n_0+j+1} (j = 1, 2, \ldots, T_{k_0} - 1)$, we have

$$\Sigma_0 \equiv \sum_{j=1}^{t_0} f(J_{n_0+j})/ (n_0 + j) < \sum_{j=1}^{t_0} A_{*n_0+j}/j.$$ 

Hence, if $t_0 < M_{k_0}$, it follows from (a) that $\Sigma_0 < e_{k_0}$. Now consider the case $t_0 > M_{k_0}$. By the second inequality in (b), we have $\Sigma_0 < 4$; furthermore, if $J_{n_0+i}, J_{n_0+i+1}$ (p < i < q) are the $J_n$'s that equal one of the intervals $I_{*n_0+j} (j = 1, 2, \ldots, T_k)$ for a fixed $k < k_0$, then since $A_{*k_0+j} > A_{*k_0+j+1} (j = 1, 2, \ldots, T_k)$ we have

$$\sum_{j=p}^{q} f(J_{n_0+i})/ (n_0 + i + j) < \sum_{j=1}^{q-p+1} A_{*k_0+j}/ (M_{k_0} + j) < e_k,$$

where the last inequality follows from (c) and the fact that $M_{k_0} > M_{k+1}$ for $k < k_0$. Consequently,

$$\sum f(J_n)/n < 4 + \sum_{k=1}^{\infty} e_k = 5,$$

and $f \in OHBV$.

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REFERENCES


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