ON THE NOTE OF C. L. BELNA

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Abstract. The space of functions of ordered harmonic bounded variation (OHBV) has been shown by Belna to contain the space of functions of harmonic bounded variation (HBV) properly. OHBV is a Banach space and HBV is a first category subset. The ordered harmonic variation has continuity properties quite different from those of the harmonic variation. The relationship of these classes to the everywhere convergence of Fourier series is discussed.

In the previous note C. Belna has shown that the inclusion \( \text{OHBV} \supset \text{HBV} \) is proper \([2]\). Here we discuss the implications of that fact for various problems on every where convergence of Fourier series and we will use Belna’s example to show an important respect in which the ordered harmonic variation behaves differently from the harmonic variation. We will also show that with a suitable norm \( \text{OHBV} \) is a Banach space and \( \text{HBV} \) is a first category subset.

With C. Goffman \([3]\), \([4]\), and \([5]\) we characterized the functions whose Fourier series converge everywhere for every change of variable. A right system of intervals at a point \( x \) is a collection of intervals \( \{I_{k,n}\} \), \( k = 1, 2, \ldots, n = 1, \ldots, n_k \), such that for each \( k \) and \( n \), \( I_{k,n+1} \) is to the right of \( I_{k,n} \) and for each \( \delta > 0 \) there is an \( N \) such that \( I_{k,n} \subseteq (x, x + \delta) \) for \( k > N \). A left system is analogously defined. Let \( f(b) - f(a) = f([a, b]) \). A function is said to be regulated if it has right and left limits at each point. A regulated function \( f \) has a Fourier series which converges everywhere for every change of variable if and only if \( \sum_{n=1}^\infty f(I_{k,n})/n \to 0 \) as \( k \to \infty \) for every right and left system at each point. The class of functions satisfying this property was called \( \text{GW} \) by A. Baernstein \([1]\) in the paper in which we characterized the continuous functions (\( \text{UGW} \)) for which the Fourier series converges uniformly for every change of variable. The condition for \( \text{UGW} \) is obtained from that for \( \text{GW} \) by adding the proviso that \( \sum_{n=1}^\infty f(I_{k,n})/(n_k - n + 1) \to 0 \).

The form of the conditions for \( \text{GW} \) and \( \text{UGW} \) clearly suggests that one look at ordered collections of intervals of the type that appear in the definition of \( \text{OHBV} \), but our results \([6]\) and \([8]\) on the convergence of Fourier series of functions of generalized bounded variation seem to require the consideration of all collections of intervals instead. The basic result here is that Fourier series of a function in \( \text{HBV} \) converges everywhere and converges uniformly on any closed interval of points of continuity. Thus \( \text{GW} \supset \text{HBV} \) and \( \text{UGW} \supset \text{HBV} \); \( X \) will denote the continuous functions in class \( X \).

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1. If a function \( f \notin \text{OHBV} \) on an interval \([a, b]\), it is easy to see that it is not in \( \text{OHBV} \) on one of the intervals \([a, (a + b)/2]\) and \([(a + b)/2, b]\). Continuing this process we can produce a point \( p \) in \([a, b]\) such that \( f \notin \text{OHBV} \) on any neighborhood of \( p \). It is elementary that on one side of \( p \) we must have \( f \notin \text{OHBV} \) on every interval ending at \( p \). Thus \( f \notin \text{UGW} \) and we have shown that \( \text{UGW} \subset \text{OHBV} \).

If we examine the example of Belna we see that the first part of condition (b) implies that

\[
\sum_{j=1}^{n} f(I_n^*)/j > 1.
\]

Choose \( I_n = [a_n, b_n] \) so that \( b_n \to 0 \) as \( k \to \infty \). We see then that \( f \notin \text{GW} \). We summarize these facts in the following statement.

**Proposition 1.**

(a) \( \text{OHBV}_C \supseteq \text{UGW} \supseteq \text{HBV}_C \).

(b) \( \text{OHBV} \cap \text{GW} \neq \emptyset \).

An interesting problem here is still unresolved.

**Question.** Is \( \text{GW} = \text{HBV} \) and \( \text{UGW} = \text{HBV}_C \)?

2. If \( f \in \text{HBV} \), then the harmonic variation of \( f \) on an interval \([a, b]\), \( V_h(f; [a, b]) \), is the supremum of sums \( \sum |f(I_n)|/n \) where the \( I_n \) are nonoverlapping intervals in \([a, b]\). A corresponding point function \( V_h(x) = V_h(f; [a, x]) \) is defined. It has been shown that \( V_h(x) \) is right (left) continuous at \( x_0 \) if and only if \( f \) is right (left) continuous there \([7], [8]\). We may define an ordered harmonic variation \( V_{oh}(f; [a, b]) \) in an analogous manner by adding the requirement of order to \( \{I_n\} \).

It is not difficult to see that the continuity of \( V_{oh}(x) = V_{oh}(f; [a, x]) \) does not follow from the continuity of \( f \).

Suppose in the example of Belna that we require \( a_n \to 1/2 \). Then \( f \equiv 0 \) on \([0, 1/2]\) and is continuous at 1/2. However \( V_{oh}(x) = 0 \) for \( x < 1/2 \), but, for \( x > 1/2 \), \( V_{oh}(x) > 1 \) by the first part of condition (b).

3. Let us consider \( \text{OHBV} \) on \([a, b]\) and let \( V(f) = V_{oh}(f; [a, b]) \). We have the following result.

**Proposition 2.** \( \text{OHBV} \) is a Banach space with norm \( \|f\| = |f(a)| + V(f) \).

It is clear that \( \text{OHBV} \) is a linear space and that \( \|f\| \) is a norm. Convergence in this norm implies uniform convergence. If \( g \) is the pointwise limit of a norm convergent sequence \( \{f_n\} \), then given \( \epsilon > 0 \), there is an \( N \) such that \( n, m > N \) imply \( \sum |f_n(I_k) - f_m(I_k)|/k < \epsilon \) for every ordered finite sequence of nonoverlapping intervals \( I_k \). Then

\[
\sum |g(I_k)|/k < \sum |f_m(I_k)|/k + \epsilon < V(f_m) + \epsilon,
\]

implying that \( g \in \text{OHBV} \). Further,

\[
V(g - f_n) < \epsilon + \sum |g(I_k) - f_n(I_k)|/k
\]
for some \( \{I_k\} \). For \( n > N \) and \( m_0 > N \) so large that 
\[ \sum |g(I_k) - f_{m_0}(I_k)|/k < \epsilon \]
we have
\[ V(g - f_n) < \epsilon + \sum |f_n(I_k) - f_{m_0}(I_k)|/k + \sum |f_{m_0}(I_k) - g(I_k)|/k < 3\epsilon, \]
and so \( ||g - f_n|| \to 0 \).

We will now show

**Proposition 3.** \( HBV \) is a first category subset of \( OHBV \).

Consider the construction of Belna. For any function \( g \) define
\[ F_m(g) = \sum_{n=1}^{\sigma_m+1} g(I_n^*)/n, \quad m = 1, 2, \ldots. \]
For any \( m \), \( F_m \) is a continuous linear functional on \( OHBV \), since it is clear that 
\[ |F_m(g)| < m||g||. \]
However
\[ F_m(f) = \sum_{k=1}^{m} \sum_{j=1}^{T_k} A_{\sigma_k+j}/(\sigma_k + j) > m \]
by condition (b) part one of Belna's construction. Thus, by the Banach-Steinhaus theorem, the set of \( g \) for which \( \{F_m(g)\} \) is bounded is of first category. For \( g \in HBV \), \( \{F_m(g)\} \) is bounded by the harmonic variation of \( g \), implying that \( HBV \) is of first category.

**References**


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