MAD FAMILIES AND ULTRAFILTERS

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Abstract. For each almost disjoint family \( X \) let \( F(X) = \{a \subseteq \omega: \text{card}\{s \in X: s \setminus a \text{ is finite}\} = 2^\omega\} \), \( I(X) = \{a \subseteq \omega: \text{card}\{s \in X: \text{card}(s \cap a) = \omega\} = 2^\omega\} \). Assuming \( P(2^\omega) \) we show that for each nonprincipal ultrafilter \( p \) there exist a maximal almost disjoint family \( X \) and an almost disjoint family \( Y \) with \( F(X) = I(Y) = p \).

1. Introduction. We refer the reader to [2] for unexplained notions. Let \( A \) be a set; \( \mathcal{P}(A) \) denotes the power set of \( A \) and \( \text{card} A \) denotes the cardinality of \( A \). \( \text{Fin} \) denotes the set of finite subsets of \( \omega \). For \( a, b \in \mathcal{P}(A) \) we write \( a \subseteq b \) if \( a \setminus b \) is finite and we write \( a = b \) if \( a \subseteq b \) and \( b \subseteq a \).

Let \( X \subseteq \mathcal{P}(\omega) \setminus \text{Fin} \). \( X \) has the fip (finite intersection property) if for any finite subset \( S \) of \( X \), \( \bigcap S \) is infinite. \( X \) is almost disjoint if (i) for \( a, b \in X \) with \( a \neq b \), \( a \cap b \in \text{Fin} \) and (ii) for any finite subset \( S \) of \( X \), \( \omega \setminus \bigcup S \) is infinite. \( X \) is called mad family if it is a maximal almost disjoint family and \( X \) is called ad family if it is an almost disjoint family.

Let \( P(2^\omega) \) be the following proposition (considered by Rothberger [5]):

If \( F \subseteq \mathcal{P}(\omega) \) has the fip and \( \text{card} F < 2^\omega \) then there is \( d \in \mathcal{P}(\omega) \setminus \text{Fin} \) with \( a \subseteq b \) for each \( b \in F \).

The proposition \( P(2^\omega) \) is weaker than Martin’s axiom (see [4]).

For \( X \) an ad family we set

\[
F(X) = \{a \subseteq \omega: \text{card}\{s \in X: s \subseteq b\} = 2^\omega\};
I(X) = \{a \subseteq \omega: \text{card}\{s \in X: \text{card}(s \cap a) = \omega\} = 2^\omega\}.
\]

Then for each ad family \( X \), \( F(X) \subseteq I(X) \); for \( X \) a mad family, \( I(X) = \{a \subseteq \omega: \text{for each finite subset } S \text{ of } X, \text{card}(a \setminus \bigcup S) = \omega\} \). We show:

**Theorem 1.** Assume \( P(2^\omega) \). Then for any nonprincipal ultrafilter \( p \) on \( \omega \) there exists a mad family \( X \) with \( F(X) = p \).

**Theorem 2.** Assume \( P(2^\omega) \). Then for any nonprincipal ultrafilter \( p \) on \( \omega \) there exists an ad family \( X \) with \( I(X) = p \).

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2. Proof of Theorems 1 and 2. Let \( p \) be any nonprincipal ultrafilter on \( \omega \), let \( \{ a_i : i < 2^\omega \} \) be an enumeration of \( p \) such that for each \( b \in p \) we have \( \text{card}\{ i < 2^\omega : b = a_i \} = 2^\omega \) and let \( \{ b_i : i < 2^\omega \} \) be an enumeration of \( \{ b \subseteq \omega : b \notin p, \text{card} \, b = \omega \} \). Let \( A_k = \{ a_i : i < k \} \), \( B_k = \{ b_i : i < k \} \). We construct increasing sequences \( \{ X_i : i < 2^\omega \} \), \( \{ Y_i : i < 2^\omega \} \) of almost disjoint sets such that for each \( i < 2^\omega \):

(i) \( \text{card} \, X_i < 2^\omega \) and \( \text{card} \, Y_i < 2^\omega \);
(ii) \( X_i \cup Y_i \cap p = \emptyset \);
(iii) \( X_i \cap Y_i = \emptyset \);
(iv) there is \( c \in X_{i+1} \setminus X_i \) with \( c \subseteq a_i \);
(v) there is \( d \in Y_{i+1} \) with \( \text{card} \, (d \cap b_i) = \omega \);
(vi) for \( i < k < 2^\omega \), if \( c \in X_k \setminus X_i \), then \( \text{card} \, (c \cap b_j) < \omega \);
(vii) for \( i < k < 2^\omega \), if \( d \in Y_k \setminus Y_i \), then \( \text{card} \, (d \cap b_j) = \omega \).

Let \( X = \bigcup \{ X_i : i < 2^\omega \} \), \( Y = \bigcup \{ Y_i : i < 2^\omega \} \). Then \( X \) is an ad family and (v) implies that \( X \cup Y \) is a mad family. (iv) implies that for each \( a \in p \), \( a \in F(X) \) and \( a \in F(X \cup Y) \). (vi) implies that for each \( a \subseteq \omega \) with \( a \notin p \), \( a \notin I(X) \). (vii) implies that for each \( a \subseteq \omega \) with \( a \notin p \), \( a \notin F(X \cup Y) \). Thus \( I(X) = F(X \cup Y) = p \).

Now we describe the construction of the \( X_i \) and \( Y_i \). We set \( X_0 = Y_0 = \emptyset \).

Assume \( i < 2^\omega \) and for each \( k < i \), \( X_k \) and \( Y_k \) are constructed. For \( i \) a limit ordinal we set \( X_i = \bigcup \{ X_k : k < i \} \), \( Y_i = \bigcup \{ Y_k : k < i \} \).

Now let \( i \) be a successor ordinal, \( i = k + 1 \). Let \( S = A_i \cup \{ \omega \setminus b : b \in B_i \} \) \( \cup \{ \omega \setminus x : x \in X_k \} \). Then \( S \) has the fip and \( \text{card} \, S < 2^\omega \). \( P(2^\omega) \) implies that there is \( a \subseteq \omega \) with \( a \setminus s \in \text{Fin} \) for each \( s \in S \). Let \( a^* \subseteq a \cap a_i \) be such that \( a^* \notin p \) and \( \text{card} \, a^* = \omega \). Then we set \( X_i = X_k \cup \{ a^* \} \). Assume there is \( s \in X_i \cup Y_k \) with \( \text{card} \, (s \cap b_j) = \omega \). Then we set \( Y_i = Y_k \). Assume now that no such \( s \) exists. Let \( T = A_i \cup \{ \omega \setminus b : b \in B_i \} \) \( \cup \{ \omega \setminus x : x \in X_i \} \). Then \( T \) has the fip and \( \text{card} \, T < 2^\omega \). \( P(2^\omega) \) implies that there is \( c \subseteq \omega \) with \( c \setminus s \in \text{Fin} \) for each \( s \in T \). Let \( c^* \subseteq c \cap a \) be such that \( c^* \notin p \) and \( \text{card} \, c^* = \omega \). Then we set \( Y_i = Y_k \cup \{ c^* \cup b_j \} \). It is now easy to see that (i)-(vii) are satisfied.

3. Topological consequences. Let \( N \) be the discrete countable space and let \( \beta N \) be the Stone-Čech compactification of \( N \). Then \( \beta N \setminus N \) can be represented by the set of all nonprincipal ultrafilters over \( \omega \) and the topology generated by the following basis \( \mathcal{A} \): For each \( a \subseteq \omega \) let \( \hat{a} = \{ p \in \beta N \setminus N : a \in p \} \) and \( \mathcal{A} = \{ \hat{a} : a \subseteq \omega \} \). Then \( \hat{a} \supseteq \hat{b} \) iff \( b \subseteq^* a \). Then Theorems 1 and 2 can be reformulated as follows:

**Theorem 1'.** Assume \( P(2^\omega) \). Then for each \( p \in \beta N \setminus N \) there is a dense system \( \mathcal{U}_p \) of open sets such that for each \( a \subseteq \omega \), \( a \in p \) iff \( \text{card} \{ U \in \mathcal{U}_p : U \subseteq \hat{a} \} = 2^\omega \).

**Theorem 2'.** Assume \( P(2^\omega) \). Then for each \( p \in \beta N \setminus N \) there is a system \( \mathcal{U}_p \) of open sets such that for each \( a \subseteq \omega \), \( a \in p \) iff \( \text{card} \{ U \in \mathcal{U}_p : U \cap \hat{a} \neq \emptyset \} = 2^\omega \).

If \( p \in \beta N \setminus N \) is a \( 2^\omega \)-point if there is a family \( \{ U_i : i < 2^\omega \} \) of pairwise disjoint open sets with \( p \in (\text{cl}\, \beta N U_i) \setminus N \). We can use Theorem 1 to derive the following theorem of Hindman [3] (Hindman used CH but there is little difficulty adapting his proof to \( P(2^\omega) \)):
Theorem 3. Assume $P(2^{\omega})$. Then each $p \in \beta N \setminus N$ is a $2^{\omega}$-point.

Proof. Let $X = \{c_i : i < 2^{\omega}\}$ be a mad family with $F(X) = p$. For each $i < 2^{\omega}$ choose an ad family $\{d_{ik} : k < 2^{\omega}\}$ with $d_{ik} \subseteq c_i$ for each $k < 2^{\omega}$. For $k < 2^{\omega}$ let

$$U_k = \bigcup \{d_{ik} : i < 2^{\omega}\}.$$

Then the $U_k$ are pairwise disjoint open sets and $p$ is in the closure of each $U_k$.

Remark. Balcar and Vojtaš [1] proved Theorem 3 without any set-theoretical assumption. It is also unknown whether Theorem 1 holds without any set-theoretical assumption.

4. Applications to superatomic Boolean algebras. Let $\mathfrak{A}$ be a Boolean algebra. $a \in |\mathfrak{A}|$ is an atom if $a \neq 0$ and for each $b \in |\mathfrak{A}|$, $a \cap b = a$ or $a \cap b = 0$. $\mathfrak{A}$ is atomic if for each $b \in |\mathfrak{A}|$ there is an atom $a$ with $a < b$. $\mathfrak{A}$ is superatomic if each homomorphic image of $\mathfrak{A}$ is atomic. $2$ denotes the two-element Boolean algebra, $\text{Pow}(\omega)$ denotes the power set Boolean algebra over $\omega$. For $A \subseteq \text{Pow}(\omega)$ let $\text{Pow}(\omega)[A]$ denote the subalgebra of $\text{Pow}(\omega)$ generated by $A \cup \omega$. For each Boolean algebra $\mathfrak{A}$, $\mathfrak{A}^{(1)}$ denotes $\mathfrak{A}$ factorized by the ideal generated by the atoms and for each $k \in \omega$ we set $\mathfrak{A}^{(k+1)} = (\mathfrak{A}^{(k)})^{(1)}$. If $X$ is a mad family then $\text{Pow}(\omega)[X]$ is a superatomic Boolean algebra whose set of atoms is $\omega$ and $(\text{Pow}(\omega)[X])^{(2)} \approx 2$.

Theorem 4. Assume $P(2^{\omega})$. Then there are $2^{2^{\omega}}$ nonisomorphic superatomic Boolean algebras $\mathfrak{A}$ whose set of atoms is $\omega$ and with $\mathfrak{A}^{(2)} \approx 2$.

Proof. Let $\mathfrak{X}$ be the class of all mad families $X$ such that $F(X)$ is a nonprincipal ultrafilter. Let $X, Y \in \mathfrak{X}$. $X$ and $Y$ are called equivalent if there are $a \in X, b \in Y$ and a one-one function $f$ from a onto $b$ such that for each $s \in X$ with $s \subseteq a$ there is $t \in Y$ with $f[s] = f[t].$ That means, $X$ and $Y$ are equivalent iff $F(X)$ and $F(Y)$ are equivalent with respect to the Rudin-Keisler order of ultrafilters. Now there are $2^{2^{\omega}}$ nonprincipal ultrafilters on $\omega$ and each equivalence class with respect to the Rudin-Keisler order contains $2^{\omega}$ ultrafilters. Let $\mathfrak{Y} \subseteq \mathfrak{X}$ be such that card $\mathfrak{Y} = 2^{2^{\omega}}$ and the elements of $\mathfrak{Y}$ are pairwise nonequivalent. Let

$$\mathfrak{F} = \{\text{Pow}(\omega)[X] : X \in \mathfrak{Y}\}.$$

Then $\mathfrak{F}$ is the desired class of superatomic Boolean algebras.

Added in Proof. As I was informed by Baumgartner, it is impossible to prove Theorem 1 without any set-theoretical assumption.

References


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