MARTIN'S AXIOM IMPLIES THAT DE CAUX'S SPACE IS COUNTABLY METACOMPACT

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ABSTRACT. De Caux defined a space $S(\mathcal{E})$ and, assuming $\clubsuit$, showed that $S(\mathcal{E})$ is normal but not countably metacompact. We assume $\text{MA}_{\omega_1}$ and show that $S(\mathcal{E})$ is countably metacompact.

In [dC] Peter de Caux constructed a Dowker space assuming $\clubsuit$. (For a definition of $\clubsuit$ and review of the Dowker space problem, see [R].) Peter Nyikos [N] asked whether it was consistent with $\text{MA} + \neg \text{CH}$ that de Caux's space be a Dowker space. Here we answer negatively by proving the title. (Martin's Axiom in the title means $\text{MA}_{\omega_1}$.)

DEFINITION. A ladder system, $\mathcal{E}$, is a sequence $(L_\lambda: \lambda \in \omega_1 \cap \text{LIM})$ such that each $L_\lambda$ is a cofinal subset of $\lambda$ with order type $\omega$.

DEFINITION. Given a ladder system $\mathcal{E}$, we define a space $S(\mathcal{E})$ as follows. The point set of $S(\mathcal{E})$ is $\omega \times \omega_1$. We define a topology on $\omega \times \omega_1$ by defining a weak base $\mathcal{B}_x$ at each point $x$. A set $U$ will be open iff $x \in U$ implies there is $B \in \mathcal{B}_x$, $B \subset U$. If $x$ is of the form $(0, \alpha)$ or $(n, \alpha + 1)$, the only element of $\mathcal{B}_x$ is $\{x\}$. If $x$ is of the form $(n + 1, \lambda)$, where $\lambda \in \text{LIM}$, then $\mathcal{B}_x$ is the countable family of sets of the form $\{(n + 1, \lambda)\} \cup \{(n) \times (S_\lambda - e)\}$, where $e$ is a finite set.

De Caux assumed $\clubsuit$ to get the existence of a ladder system $\mathcal{E}$ with special properties, and used the special properties to show that $S(\mathcal{E})$ is a Dowker space. Nyikos calls any space of the form $S(\mathcal{E})$ a de Caux's Litmus strip space. Clearly, one can construct $S(\mathcal{E})$ without assuming $\clubsuit$. Further, the properties of $S(\mathcal{E})$ could vary with $\mathcal{E}$. It is not hard to define $\mathcal{E}$ so that $S(\mathcal{E})$ is a $\sigma$-discrete Moore space, for example. What we are asserting in this paper is that assuming $\text{MA} + \neg \text{CH}$, all spaces of the form $S(\mathcal{E})$ are countably metacompact.

DEFINITION. A space is countably metacompact iff whenever $(H_\alpha)_{\alpha<\omega}$ is a decreasing sequence of closed sets with empty intersection then there is a sequence $(U_\alpha)_{\alpha<\omega}$ of open sets with empty intersection satisfying $U_\alpha \supset H_\alpha$.

DEFINITION. A poset $P$ has property $K$ iff whenever $W$ is an uncountable subset of $P$ there is an uncountable subset $W'$ of $W$ of compatible elements. The product of property $K$ posets has property $K$, a fortiori, ccc.

DEFINITION. For $A$ a set $[A]^2$ is the set of two element subsets of $A$. If $A$ is totally ordered by $<$, we may think of $[A]^2$ as $\{(a, b) \in A \times A: a < b\}$. A theorem of...
Dushnik and Miller\textsuperscript{2} asserts that wherever $(A, <)$ has order type $\omega_1$ and $f: [A]^2 \to \{0, 1\}$ then either there is a subset, $S_0$, of $A$ of order type $\omega_1$ such that $f(a, b) = 0$ for all $(a, b) \in [S_0]^2$ [we will abbreviate $f(a, b)$ by $f(a, b)$] or there is a subset, $S_1$, of order type $\omega + 1$ such that $f(a, b) = 1$ for all $(a, b) \in [S_1]^2$. We abbreviate this theorem by $\omega_1 \to (\omega_1, \omega + 1)^2$.

Now, assume we are given a space $S(\mathcal{E})$, and a decreasing sequence $(H_n)_{n \in \omega}$ of closed sets with empty intersection. We will use $\operatorname{MA} + \neg \operatorname{CH}$ to define a sequence $(U_n)_{n \in \omega}$ of open sets with empty intersection satisfying $U_n \supset H_n$.

We define $P_n$, a poset of approximations to $U_n$. Our plan is to define

$$U_n = \bigcup \{ c_p: p \in G \} = \bigcup \{ \bigcup \text{ range } b_p: p \in G \}.$$ 

Let $P_n$ be the set of triples $p = (a_p, c_p, b_p)$, usually abbreviated $(a, c, b)$, satisfying

- $a$ is a finite subset of $S(\mathcal{E}) - H_n$,
- $c$ is a finite subset of $S(\mathcal{E})$,
- $b$ is a function with domain $c$, $b(x) \in \mathcal{B}_x$, $\bigcap c \neq \emptyset$.

We claim that $P$ has property $K$. Let $W$ be an uncountable subset of $P$. First we apply the $\Delta$-system lemma to $\{ a_p: p \in W \}$ to get an uncountable $W_1 \subset W$ such that $\{ a_p: p \in W_1 \}$ is a $\Delta$-system with root $r_a$. Similarly obtain an uncountable $W_2 \subset W_1$ such that $\{ c_p: p \in W_2 \}$ is a $\Delta$-system with root $r_c$. Next, find an uncountable $W_3 \subset W_2$ such that for all $p, q \in W_3$, $b_p | r_c = b_q | r_c$ and card $c_p = k = \text{card } c_q$. Now we define $W_4 = \{ p(\beta): \beta < \omega_1 \}$ by induction on $\beta$ so that the

- unfixed part of $p(\beta)$ is strictly above the sup of the unfixed parts of $p(\beta')$, $\beta' < \beta$.

Precisely, for $x = (n, a) \in S(\mathcal{E})$ define $h(x) = x$; define $h(a) = \max \{ h(x): x \in a \}$, $h(c) = \max \{ h(x): x \in c \}$, $h(p) = \max \{ h(a_p), h(c_p) \}$.

Define $p(\beta)$ so that if $x \in c_p(\beta) \cup c_p(\beta)$ and $h(x) < \sup \{ h(p(\beta')): \beta' < \beta \}$, then $x \in r_a \cup r_c$.

All the above refining has achieved: If $\beta' < \beta$, and $p(\beta)$ and $p(\beta')$ are incompatible, then there is $x \in c_p(\beta) - r_c$ such that $b_p(\beta)(x) \cap b_p(\beta') = \emptyset$. List each $c_p(\beta)$ as $(x(\beta, j))_{j < k}$. We define a function $f_0: [\omega_1]^2 \to \{0, 1\}$ by $f(\beta', \beta) = 0$ iff $b_p(\beta)(x(\beta, 0)) \cap a_p(\beta) = \emptyset$. We apply $\omega_1 \to (\omega_1, \omega + 1)^2$ to get either $S_0$ or $S_1$.

We claim that we get $S_0$ of order type $\omega_1$. Aiming for a contradiction, assume $S_1 \subset W_4$ has order type $\omega + 1$ and $f(\beta', \beta) = 1$ for all $(\beta', \beta) \in [S_1]^2$. Let $(\gamma_n)_{n < \omega}$ enumerate $S_1$ in increasing order. Let $x(\gamma_n, 0) = (m + 1, \lambda)$. Let $(m, \delta_n) \in b_p(\gamma_n) \cap c_p(\gamma_n)$ Then $(\delta_n)_{n < \omega}$ is an increasing sequence of elements of $L_\lambda$ whose sup is less than $\lambda$. Contradiction.

Define $f_1: [S_0]^2 \to \{0, 1\}$ as we did $f_0$ with $x(\beta, 0)$ replaced by $x(\beta, 1)$. The same argument gives that we get a new $S_0$ (rather than $S_1$). We repeat, applying $\omega_1 \to (\omega_1, \omega + 1)^2$ $k$ times, to get $S^* \subset \omega_1$ of order type $\omega_1$; $\{ p(\beta): \beta \in S^* \}$ is an

\textsuperscript{2}Note exactly. Some unidentified person noted that some proofs of the Dushnik-Miller theorem actually give this slightly stronger result. See [W, Theorem 7.4.1]; [F, Theorem 4.5]. It is also a special case of Corollary 1 to Theorem 3.4, [ER].
uncountable subset of $W$ of compatible elements. We have shown that $P_n$ has property $K$.

Let $P$ be the product poset $\prod_n P_n$. We consider $P$ to be the set of functions $f$ with domain $\omega$ such that $f(n) \in P_n$ and $(n \in \omega : f(n) \neq (\emptyset, \emptyset, \emptyset))$ is finite; $f' < f$ iff for all $n, f'(n) < f(n)$. The following subsets of $P$ are dense: for all $x \in S(\mathcal{L})$,

$$D_x = \{ f \in P : \exists n x \in a_{f(n)} \}$$

for all $n \in \omega$, $x \in H_n$

$$D_{x,n} = \{ f \in P : x \in c_{f(n)} \}$$

for all $n \in \omega$, $x \in S(\mathcal{L})$

$$D_{x,n}' = \{ f \in P : x \in c_{f(n)} \cup a_{f(n)} \}.$$  

Since there are $\omega_1$ of the above dense subsets, by $\text{MA} + \neg \text{CH}$ there is a filter $G$ on $P$ meeting all of them. For each $n \in \omega$, define $U_n = \bigcup \{ c_{f(n)} : f \in G \}$. Then $(U_n)_{n \in \omega}$ is a sequence of open subsets with empty intersection satisfying $U_n \supset H_n$. (The first family of dense sets ensures that $(\sim)^{\omega_1} \cap U = \emptyset$; the second that $U_n \supset H_n$; the third that $U_n$ is open.) We have shown that $S(\mathcal{L})$ is countably metacompact.

**Addenda.** 1. A simple modification of the proof shows that, assuming $\text{MA}_{\omega_1}$, every subset of $S(\mathcal{L})$ is $G_\delta$.

2. P. Nyikos has also asked whether the following statement, $\Sigma$, is a theorem of ZFC.

"There is a ladder system $\mathcal{L} = (L_\lambda : \lambda \in \omega_1 \cap \text{LIM})$ such that whenever $x$ is a stationary subset of $\omega_1$, then there is $\lambda < \omega_1$ such that $x \cap L_\lambda$ is infinite."

Negative answers have been given by Komjáth, Kunen, and the author. Kunen showed that $\text{MA}_{\omega_1} \rightarrow \neg \Sigma$, using the product of $\omega$ copies of Wage's poset $[W]$. By a density argument, the union of the countably many "generic" sets is $\omega_1$, so at least one must be stationary. The author's negative answer is now a technique looking for an application. The key idea is that the following statement is consistent with ZFC.

"$\text{MA} + 2^\omega = \omega_3 + \text{there is a family } \mathcal{C} \text{ of club subsets of } \omega_1 \text{ satisfying}

1. \mathcal{C} \text{ has cardinality } \omega_2.
2. \text{If } C \text{ is club in } \omega_1, \text{ then there is } C' \in \mathcal{C}, C' \subset C."

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