PERIODIC AND LIMIT ORBITS
AND THE DEPTH OF THE CENTER
FOR PIECEWISE MONOTONE INTERVAL MAPS

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Abstract. For a piecewise monotone map of the interval: (a) the nonwandering points outside the closure of the periodic points are isolated in the nonwandering set; (b) the forward orbit of any such point misses all turning points; (c) the depth of the center is at most 2; and (d) all \( \omega \)-limit points belong to the closure of the periodic points.

Let \( f: I \to I \) denote a piecewise-monotone map of the interval: \( f \) is continuous, and strictly increasing or decreasing on each interval of a finite partition \( I = [c_0, c_1] \cup \cdots \cup [c_{k-1}, c_k] \). The points \( c_i \) \( (i = 1, \ldots, k - 1) \) forming the coarsest such partition are the turning points of \( f \). Denote the nonwandering set of \( f \) by \( \beta \) and the set of periodic points by \( P \). In general, \( \Omega \neq \overline{P} \) [B], [Y], although generically \( \Omega = \overline{P} \) [Y]. However, \( \overline{P} \) always contains the recurrent (Poisson stable) orbits [Y], even when \( f \) is just continuous [CH]. An analysis of the points in \( \Omega - \overline{P} \) is carried out in [Y]; in [CH] it is shown (for any continuous \( f \)) that \( \Omega - \overline{P} \) is nowhere dense in \( I \). In this note, we sharpen the analysis in [Y] to prove the following

Theorem. If \( f: I \to I \) is piecewise monotone, then

(a) each point of \( \Omega - \overline{P} \) is isolated in \( \Omega \);
(b) if \( x \in \Omega - \overline{P} \), then \( f^n(x) \) is not a turning point of \( f \), for any \( n > 0 \);
(c) \( \Omega(f|\Omega) = \overline{P} \);
(d) for any \( z \in I \), \( w(z) \subset \overline{P} \) \( (\omega = \omega\text{-limit set}) \).

Statement (b) for \( n = 0 \) is shown in [Y]. To put statement (c) in perspective, we recall the definition of the Birkhoff center of a dynamical system: let \( \Omega^0 \) be the full phase space (\( I \) in our case); define \( \Omega^{i+1} = \Omega(f|\Omega^i) \), and for limit ordinals define \( \Omega^\omega = \bigcap \{ \Omega^i \mid j < i \} \). For some ordinal \( \delta \), \( \Omega^\delta = \Omega^{\delta+1} = \cdots \); this set is called the Birkhoff center, and the least such ordinal \( \delta \) is called the depth of the center. It has been shown that \( \delta < 2 \) for flows on orientable compact surfaces [ST] and \( \delta < 3 \) on nonorientable ones [T], [N1]; on the other hand, \( \delta \) is arbitrary for flows on manifolds of higher dimension and on nonorientable open surfaces [N2] (see references there to earlier work of Mäier). Since \( \overline{P} \subset \Omega^1 \) for all \( i \), our statement (c) is the equivalent of \( \delta < 2 \) for piecewise monotone interval maps.

All the statements of the theorem are easy corollaries of the following technical
PROPOSITION. If \( f: I \to I \) is a piecewise monotone map and \( x \in \Omega - \overline{P} \), then for some neighborhood of \( x \), \( J = J_- \cup J_+ \), where \( J_\pm \) are closed intervals abutting at \( x \), and

(i) \( f^nJ_+ \cap J = \emptyset \) for all \( n > 0 \);
(ii) \( f^nJ_- \cap J \subset J_+ \) for all \( n > 0 \).

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PROOF OF THEOREM. To see that the proposition implies (a), (c), and (d) of the theorem, pick \( x \in \Omega - P \): we note that (a) \( \text{int } J_+ \) and \( \text{int } J_- \) are both wandering intervals, so \( \Omega \cap \text{int } J = \{x\} \), (c) since \( x \) does not return to \( J \), \( \Omega \cap \text{int } J \) is wandering for \( f|\Omega \), and (d) any particular orbit enters \( J \) at most twice (once in \( J_- \), once in \( J_+ \)) so that \( J \) intersects no \( \omega \)-limit sets.

To see (b), note that if \( f^n(x) \) is a turning point for \( f \), then \( f^{n+1} \) folds some interval \((x - \epsilon, x + \epsilon)\), which we can assume is contained in \( J \); to every \( y \in (x - \epsilon, x) \) there corresponds \( z \in (x, x + \epsilon) \) such that \( f^{n+k}(y) = f^{n+k}(z) \) for all \( k > 0 \). But \( y \in J_- \) if and only if \( z \in J_+ \); since points of \( J_- \) arbitrarily near \( x \) return to \( (x - \epsilon, x + \epsilon) \) under arbitrarily high iterates of \( f \), we have \( f^{n+k}J_+ \cap f^{n+k}J_- \cap J \neq \emptyset \) for some large \( k \), contradicting (i) of the proposition. □

REMARK. The following corollaries of the intermediate value property for the real continuous functions \( F_n(x) = f^n(x) - x \) will be useful in the proof of the proposition:

(1) If \( K \subseteq I \) is a closed interval and either \( f^n(K) \subseteq K \) or \( f^n(K) \supset K \), then \( K \) contains a fixed point of \( f^n \), hence \( K \cap P \neq \emptyset \).

(2) If \( K \subseteq I \) is an interval disjoint from \( P \) and \( f^n(y) > y \) (resp. \( f^n(y) < y \)) for some \( y \in K \), then the same holds for all \( y \in K \).

PROOF OF PROPOSITION. Pick \( x \in \Omega - \overline{P} \), and let \( J = [x - \epsilon, x + \epsilon] \) be a closed neighborhood of \( x \), disjoint from \( P \):

(A) \( J \cap P = \emptyset \).

Since \( x \) is not recurrent (by results mentioned in the introduction), we can adjust \( \epsilon \) so that

(B) \( f^n x \notin J \) for any \( n > 0 \).

Lemma 1 of [Y] says that

(C) There exists \( y_k \to x \) and \( n_k \to \infty \) so that \( f^{n_k}y_k = x \).

A subsequence of the \( y_k \) belongs to one of the half-intervals \([x - \epsilon, x]\) and \([x, x + \epsilon]\): we denote this one by \( J_- \), and let \( J_+ \) be the other half of \( J \). We can assume without loss of generality that \( J_- = [x - \epsilon, x] \). By adjusting \( \epsilon \), we can assume

(D) \( f^N(x - \epsilon) = x \), and \( f^ny \neq x \) for \( n < N \) and \( y \in \text{int } J_- \).

Our first observation is

(E) **LEMA.** There exists \( \delta, 0 < \delta < \epsilon \), so that \( f^n z \notin (x - \delta, x) \) for any \( n > 0 \) and \( z \in J_+ \).
Proof of (E). For the moment, pick $0 < \delta < \epsilon$, and suppose for some $z \in J_+$ that $f'z \in (x - \delta, x)$. Then, since $f'z < z$, (A) implies that $f'x < x$ and hence by (B), $f'x < x - \epsilon$. Thus,

(E1) $f'[x, z] \supset [x - \epsilon, x - \delta]$. 

Now, for $\epsilon > 0$ fixed and $\delta > 0$ sufficiently small, one of the points $y_k$ in (C) belongs to $(x - \epsilon, x - \delta)$. Thus, $f^n[y_k] = x > y_k$; by (D), $n_k > N$, and so $f^n(x - \epsilon) = f^{n-N}(x) > x + \epsilon$ by (A) and (B). Hence, 

(E2) $f^n[x - \epsilon, y_k] \supset [x, x + \epsilon]$.

But then (E1) and (E2) imply

$$f^{1+n}[x, z] \supset f^n[x - \epsilon, x - \delta] \supset f^n[x - \epsilon, y_k] \supset [x, x + \epsilon] \supset [x, z]$$

contradicting (A). □

Note that by remark (1) above, if $f(z) = x < z$, then $n(x) < x$, contradicting (E). Thus replacing $\epsilon$ with $\delta$, we can strengthen (E) to

(\tilde{E}) $f^n J_+ \cap J_+ = \emptyset$ for all $n > 0$.

To continue, we invoke piecewise monotonicity. Recall the following notation from [Y]:

\begin{align*}
\Gamma_+(y) &= \text{clos}\{ f^n y | n > 0 \}, \\
\Gamma_-(y) &= \{ z \exists y_k \to z, n_k \to \infty \text{ with } f^n y_k = y \}.
\end{align*}

Since there are finitely many turning points for $f$, we can, by shrinking $J$ further, assume

(F) If $c$ is a turning point of $f$ whose forward and backward orbits both hit $J$, then $x \in \Gamma_+(c) \cap \Gamma_-(c)$.

By [Y], nonwandering turning points belong to $P$. Thus, by more shrinking, we can assume $J$ contains no turning points for $f$.

Next, we show that

(G) If $f^n J \cap J$ is nonempty, it is an interval, with one endpoint $x \pm \epsilon$ and the other either $x$ or $f^k(c)$, where $0 < k < n$ and $c$ is a turning point as in (F).

Proof of (G). If neither endpoint of $f^n J \cap J$ is $x \pm \epsilon$, then $f^n J \subset J$, contradicting (A). On the other hand, $x \pm \epsilon$ cannot both be endpoints of $f^n J$, since $f^n J = J$ would also contradict (A). By (E) and remark (1), no image of $x + \epsilon$ can be interior to $J$; by (B) and (D), the only image of $x - \epsilon$ interior to $J$ is $x = f^N(x - \epsilon)$.

Finally, if an endpoint of $f^n J \cap J$ is $f^n(y), y \in \text{int } J$, then $y$ is a turning point of $f^n$, so that some turning point $c$ for $f$ satisfies $c = f^{n-k}(y), f^n(y) = f^k(c), k > 0, n - k > 0$. □

(H) The left endpoint of $f^n J \cap J$ is not interior to $J_-$.

Proof of (H). If $f^n J \cap J = [d, x + \epsilon]$ and $x - \epsilon < d < x$, then $d = f^k(c)$ for $c$ some turning point, as in (F). Let $(a, b)$ be a maximal interval containing $c$ such that $f^k(a, b) \subset \text{int } J$. Unless it is an endpoint of the ambient interval $I$, each of the points $a$ and $b$ maps to $x + \epsilon$; since $f^k I \subset J$, at least one of the points $a$ and $b$ is not an endpoint of $I$.

Since $x \in \Gamma_-(c) \cap (d, x + \epsilon)$, there exist $y \in f^n J \cap J$ and $l$ such that $f^l y = c$ and hence $f^{l+k} y = d < y$. On the other hand, since $f^{l+k} x \notin J$, some point $z$ between $x$ and $y$ maps under $f^l$ to $a$ (or $b$—in any case, not an endpoint of $I$) and
hence \( f^{l+k}(z) = x + e > z \). But then \( f^{l+k}[d, x + e] \supset [d, x + e] \), contradicting (A).

A further refinement of (G) and (H) is

(I) For each \( n > 0 \), \( f^n J \cap J \) (if nonempty) is one of two types: \([x - e, d_] \) with \( d_ < x \), or \([d_+, x + e] \) with \( d_+ > x \).

Proof of (I). We know that in the second case \( d_+ > x \) or \( d_+ = x - e \). But \( d_+ = x - e \) means \( f^n J \supset J \), contradicting (A). In the first case, if \( d_ > x \) then \( J \subset f^n J \cap J_+ \); but since \( f^n J_+ \cap J_+ = \emptyset \), we must have \( J \subset f^n J_+ \), again contradicting (A).

Now, consider the finite set of turning points \( c_1, \ldots, c_i \) for \( f \) such that \( x \in \Gamma_-(c) \cap \Gamma_+(c) \), and for each \( i \), let \( d_i \) be the first forward image of \( c_i \) in \( J \). Let \( d_0 = x - e \), and set \( x - \delta = \max\{d_i < x\} \).

(J) Lemma. \( f^n J \cap (x - \delta, x) = \emptyset \) for all \( n > 0 \).

Proof of (J). Pick \( n \) the least positive integer for which \( f^n J \) intersects \((x - \delta, x)\). By (I), \( f^n J \cap J \) must have the form \([x - e, d_] \), with \( x - \delta < d_ < x \). By (G), \( d_ = f^k(c_i) \) for some \( k < n \) and \( c_i \) one of the turning points above; furthermore, \( c_i \in f^{n-k} J \). Now, if \( d_ \neq d_i \), then \( d_ = f^l d_i \) for some \( 0 < l < k \), and since \( d_i \in J \), this means \( f^n J \cap (x - \delta, x) \neq \emptyset \) with \( 0 < l < n \), contradicting the minimality of \( n \). But then \( d_ = d_i \), this time contradicting the choice of \( x - \delta \). This establishes (J).

The proposition follows easily: by further shrinking \( J \), we can assume \( J \subset (x - \delta, x) \) so that \( f^n J \cap \text{int} J_+ = \emptyset \) (in other words, \( f^n J \subset J_+ \)) for all \( n > 0 \). On the other hand, \( f^n J_+ \cap J_+ = \emptyset \) for all \( n > 0 \) by (E). These two statements give the proposition.

REFERENCES


