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CHARACTER TABLES DETERMINE ABELIAN SYLOW 2-SUBGROUPS

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ABSTRACT. A finite group has an abelian S_2 if and only if every 2-element is 2-central.

In his survey talk at the AMS Summer Institute on Finite Group Theory, Santa Cruz, Califormia, 1979, Walter Feit mentioned the following problem:

Can one read from the character table of a finite group if its Sylow p-subgroups are abelian?

The aim of this note is to show that a Sylow p-subgroup of G is abelian iff each p-element of G is p-central, provided that either $p = 2$ (Theorem 6) or G is p-solvable (Proposition 1). Thus in these two cases the answer to Feit's question is in the affirmative. The authors are not aware of finite groups satisfying one of the above-mentioned properties, but not the other. In this note we also show that the property: "a Sylow 2-subgroup of G is elementary abelian" can be read from the character table of G (Corollary 5).

In this note G denotes a finite group. Let p be a prime. The symbol S_p stands for "Sylow p-subgroup". An arbitrary Sylow p-subgroup of G will also be denoted by S_p. An element x of G is called *p-central* if its centralizer contains an S_p of G, and it is called *real* if its column in the character table of G is real. It is well known that x is real iff x is conjugate to x^{-1} in G. Finally, let $C_G^*(x) = \{g \in G | x^g = x \text{ or } x^{-1}\}$. This is a subgroup of G and $C_G^*(x) = C_G(x)$ unless x is a real element satisfying $x^2 \neq 1$, in which case $|C_G^*(x): C_G(x)| = 2$.

If G is p-solvable, we can easily prove the following proposition.

PROPOSITION 1. *Let G be a p-solvable finite group. Then S_p of G is abelian iff each p-element of G is p-central.*

PROOF. The "only if" part is trivial. So suppose that each p-element of G is p-central. By Theorem 3.3 in [3], $G = O_{p'pp'}(G)$. Now $\overline{G} = G/O_{p'}(G)$ is p-closed and still satisfies our assumption. Thus S_p of $\overline{G}$ is abelian and hence S_p of G is abelian.

From now on, we shall deal with the prime $p = 2$ and G will denote a group of even order. The following remark is trivial, but basic.

PROPOSITION 2. *A nontrivial 2-central element of G is real iff it is an involution.*

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PROOF. Every involution is real. Conversely, let $u \neq 1$ be a 2-central element of G. Then $C_G^*(u) = C_G(u)$. Thus, if u is real, it follows that u is an involution.

COROLLARY 3. *The 2-central involutions of G are determined by the character table of G.*

COROLLARY 4. *An S_2 of G is elementary abelian iff every 2-element of G is 2-central and real.*

COROLLARY 5. *The property "an S_2 of G is elementary abelian" is determined by the character table of G.*

Finally, we prove

THEOREM 6. *Let G be a finite group. Then S_2 of G is abelian if and only if each 2-element of G is 2-central.*

PROOF. The "only if" part is trivial. By Proposition 1 we may assume that G is a nonsolvable group of minimal order such that each 2-element of G is 2-central, but S_2 of G is nonabelian. Clearly $O(G) = 1$ and $O^{2'}(G) = G$. If u and v are distinct involutions in S_2, then uv is a real 2-element, hence an involution by Proposition 2. Thus $\Omega_1(S_2)$ is abelian. Moreover, it follows from our assumptions that $F(G) = O_2(G)$ is centralized by each 2-element of G, hence $O_2(G) < Z(G)$.

Let E be the central product of all subnormal quasisimple subgroups of G and let $F^* = O_2(G)E$. It is well known that $C_G(F^*) = Z(F^*)$ (see [**1**, §10]), hence $E \neq 1$. Suppose that G is quasisimple. As $\Omega_1(S_2)$ is elementary abelian and every 2-element of G is 2-central, it follows by [**2**] that G is simple with an abelian S_2, a contradiction. So each quasisimple subnormal subgroup of G is a proper subgroup of G, hence has an abelian S_2. Consequently, $F^* = O_2(G)*M_1 * \cdots * M_r$, a central product, with M_i quasisimple having abelian S_2. Thus F^* has an abelian S_2 and it suffices to show that if x is a 2-element of G, then $x \in F^*$. Suppose that $x \notin F^*$. Then x acts as an automorphism on F^*, centralizing $O_2(G)$ and permuting the M_i. However, x centralizes an S_2 of F^*, hence x normalizes each M_i. Thus we may assume that x acts as an outer automorphism of even order on some M_i, and hence also on the simple group $M = M_1/Z(M_1)$ [**1**, Lemma 10.3]. Since M is a simple group with an abelian S_2 and with an outer automorphism of even order, it follows by [**4**] that $M = PSL(2, q)$, with $q = 2^n > 2$ or $q \equiv 3$ or $5 \pmod 8$, $q > 5$. If $q = 2^n$ and the order of x as an outer automorphism is $2^k = r$, then $C_M(x) = PSL(2, 2^{n/r})$ and x does not centralize an S_2 of M, a contradiction. If $q \equiv 3$ or $5 \pmod 8$ and $q > 5$, then $Z(M_1) = 1$, and x acts on M_1 as an element of $PGL(2, q)$, hence it does not centralize an S_2 of M_1, a contradiction. The proof of Theorem 6 is complete.

COROLLARY 7. *The property "an S_2 of G is abelian" is determined by the character table of G.*

ADDED IN PROOF. The authors have been informed that ${}^2F_4(2)$ has just one class of 3-elements and its S_3 is nonabelian of order 27 and exponent 3.

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