ASYMPTOTIC PRIME DIVISORS AND ANALYTIC SPREADS

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Abstract. Let \( I \) be an ideal in a Noetherian domain \( R \), and let \( \hat{I} \) be the integral closure of \( I \). Let \( \hat{A}^*(I) = \text{Ass}(R/\hat{I}^n) \) for \( n \) large (it being known that for large \( n \) this set does not vary with \( n \)). Suppose that \( R \) satisfies the altitude formula. Then it is shown that \( P \in \hat{A}^*(I) \) if and only if height \( P = \ell(I_P) \), the analytic spread of \( I_P \).

Introduction. Let \( I \) be an ideal in a Noetherian ring. For \( n > 1 \), let \( A(n) \) be the set of prime divisors of \( I^n \), \( A(n) = \text{Ass}(R/I^n) \). A recent paper of Brodmann [1] shows that \( A(n) \) is constant for \( n \) large. In [5] that constant is denoted \( A^* = A^*(I) \). In general it is difficult to explicitly determine \( A^* \) for a given ideal \( I \), although in [5, Corollary 22] this is done for \( R \) a 2-dimensional normal domain. This paper will discuss a concept related to \( A^* \), namely \( \hat{A}^* \). Let \( \hat{I} \) denote the integral closure of the ideal \( I \), and let \( \hat{A}(n) = \text{Ass}(R/\hat{I}^n) \), the prime divisors of \( \hat{I}^n \). If height \( I > 1 \), [5, Proposition 7] shows that \( \hat{A}(n) \) is constant for large \( n \). That constant is denoted by \( \hat{A}^* = \hat{A}^*(I) \). The purpose of this paper is to characterize \( \hat{A}^* \) for any ideal \( I \) in a Noetherian domain satisfying the altitude formula. The characterization is \( P \in \hat{A}^* \) if and only if height \( P = \ell(IR_P) \), the analytic spread of \( IR_P \).

Preliminaries. Throughout this paper, \( R \) will denote a Noetherian domain, \( I \) an ideal of \( R \), and \( P \) a prime ideal of \( R \) containing \( I \). The domain \( T \) will always be \( T = R[Ix] = R + Ix + I^2x^2 + \ldots \), \( x \) an indeterminate. Since \( T \subset R[x] \), obviously the transcendence degree of \( T \) over \( R \) is 1. We will occasionally mention the form ring of \( I \), \( R/I + I/I^2 + \ldots \). Note that this is isomorphic to \( T/IT \). If \( (R, P) \) is local, we will also use the ring \( R/P + I/P + I^2/P^2 + \ldots \), which is isomorphic to \( T/PT \). Finally, \( P'' \) will be \( P + Ix + I^2x^2 + \ldots \) in \( T \).

If \( (R, P) \) is a local domain and \( I \) is an ideal of \( R \), then \( l(I) \) denotes the analytic spread of \( I \). Recall that there are various characterizations of \( l(I) \). (i) If \( R/P \) is infinite and if \( J \) is a minimal reduction of \( I \) then \( l(I) = \psi(J) \), the minimal number of generators of \( J \). (ii) \( l(I) = \text{height}(P''/PT) \). (See [7] and [8] for basics on reductions and \( l(I) \). Also by the altitude inequality (stated below) height \( P + \text{TRD}(T/R) > \text{height} P'' + \text{TRD}(P''/P) \) giving height \( P + 1 > \text{height} P'' > \text{height}(P''/PT) = l(I) \). Thus height \( P > l(I) \). (See [2] for more.)

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Let the domain $S$ be a finitely generated ring extension of $R$. Let $Q$ be prime in $S$ with $Q \cap R = P$. It is well known that $\text{height } P + \text{TRD}(S/R) > \text{height } Q + \text{TRD}(Q/P)$. Here “TRD” denotes transcendence degree and $\text{TRD}(Q/P)$ refers to the transcendence degree of $S/Q$ over $R/P$. If in fact $\text{height } P + \text{TRD}(S/R) = \text{height } Q + \text{TRD}(Q/P)$ for all such $S$ and $Q$, then $R$ is said to satisfy the altitude formula. Almost all known Noetherian domains do satisfy the altitude formula. The only known counterexamples are variations on [6, Example 2, pp. 202–205]. Thus assuming that the altitude formula holds is a minor restriction.

$\hat{A}^*$ and $(I)$. Our first lemma is essentially a restatement of [5, Proposition 18] in a more efficient manner.

**Lemma 1.** Let $R$ be a Noetherian domain which satisfies the altitude formula. Let $0 \neq I \subseteq P$ be ideals of $R$, with $P$ prime. Then $P \in \hat{A}^*$ if and only if there is a height one prime $P'$ of $T = R +Ix + I^2x^2 + \ldots$, with $P' \cap R = P$. If this is the case, then $P'$ is homogeneous.

**Proof.** Suppose first that such a $P'$ exists. As $I \subseteq P \subseteq P'$, $IT \subseteq P'$ and so in the form ring of $I$, $T/IT$, $P'/IT$ is a minimal prime. Applying the altitude formula to $R \subseteq T$ and the primes $P$ and $P'$ gives $\text{height } P + \text{TRD}(T/R) = \text{height } P' + \text{TRD}(P'/P)$, that is, $\text{height } P + 1 = 1 + 0$. Thus $\text{height } P = 0$ contradicting that $0 \neq I \subseteq P$. Therefore $P'/IT$ is relevant, as required.

Conversely, suppose that $P \in \hat{A}^*$. By [5, Proposition 18], in the form ring $T/IT$ there is a minimal prime, call it $P'/IT$, with $(P'/IT) \cap (R/I) = P/I$. To prove the lemma, we must only show that in $T$, $\text{height } P' = 1$. We go to the Rees ring $T + x^{-1}R[x^{-1}]$, and consider the prime $P' + x^{-1}R[x^{-1}]$. Since $T$, being a finitely generated extension of $R$, satisfies the altitude formula, and since $T/P' = (T + x^{-1}R[x^{-1}])/(P' + x^{-1}R[x^{-1}])$ we have height $P' = \text{height } P' + x^{-1}R[x^{-1}]$. As $P'$ is minimal over $IT$, $P' + x^{-1}R[x^{-1}]$ is minimal over $IT + x^{-1}R[x^{-1}] = x^{-1}(T + x^{-1}R[x^{-1}])$, which is a principal ideal of the Rees ring. Accordingly, $\text{height } P' = 1$.

**Corollary 2.** Let $(R, P)$ be a local domain satisfying the altitude formula. Let $I$ be an ideal of $R$. Then $P \in \hat{A}^*$ if and only if $PT$ is a height one ideal of $T$.

**Proof.** If $P \in \hat{A}^*$, pick $P'$ as in the lemma. Obviously $PT \subseteq P'$ and so height $PT = 1$. Conversely if height $PT = 1$, let $P'$ be a height one prime of $T$ containing $PT$. Thus $P \subseteq PT \subseteq P'$ and so $P' \cap R = P$. By the lemma, $P \in \hat{A}^*$.

**Theorem 3.** Let $R$ be a Noetherian domain satisfying the altitude formula. Let $I \neq 0$ be an ideal of $R$ and let $P$ be a prime containing $I$. Then $P \in \hat{A}^*$ if and only if $I(PR_p) = \text{height } P$.  

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Proof. We may assume that $R$ is local at $P$, and write $l(I)$ for $l(IR_P)$. Suppose first that height $P = l(I)$, call this $n$. Let $P''$ be the prime $P + Ix + I^2x^2 + \ldots$ of $T = R + Ix + I^2x^2 + \ldots$. Since $n = l(I) = \text{height}(P''/PT)$ in the ring $T/PT$, we have a chain of primes $P'_0 \subseteq P'_1 \subseteq \cdots \subseteq P'_n = P''$ in $T$ with $PT \subseteq P'_0$. Obviously $P'_0 \cap R = P$. In order to show that $P \in \hat{A}^*$, in view of the lemma we must only show that height $P'_0 = 1$. We apply the altitude formula to $R \subset T$ and the primes $P$ and $P''$. Since height $P = n$, $\text{TRD}(T/R) = 1$, and $T/P'' = R/P$, the altitude formula yields $n + 1 = \text{height} P''$. Now the chain $P'_0 \subseteq P'_1 \subseteq \cdots \subseteq P'_n = P''$ shows that height $P'_0 = 1$ as required.

Conversely, suppose that $P \in \hat{A}^*$, and now let $n = l(I)$. As above we have $P'_0 \subseteq P'_1 \subseteq \cdots \subseteq P'_n = P''$ with $PT \subseteq P'_0$. Since $P \in \hat{A}^*$, by Lemma 1 there is a height 1 homogeneous prime $P'$ of $T$ with $P' \cap R = P$. As $(R, P)$ is local, and $P'$ is homogeneous, $P' \subseteq P + Ix + I^2x^2 + \cdots = P''$. Let $k = \text{height}(P''/P')$. As $P \subseteq P'$, $PT \subseteq P'$. Thus $k = \text{height}(P''/P') < \text{height}(P''/PT) = l(I) = n$. That is $k < n$. As $R$ satisfies the altitude formula, $T$ is catenary [4, Corollary 2.5] and so height $P'' = \text{height}(P''/P') + \text{height} P' = k + 1$. Also height $P'' = \text{height}(P''/P'_0) + \text{height} P'_0 > n + 1$ (by the existence of the chain $P'_0 \subseteq \cdots \subseteq P'_n = P''$ and the fact that $0 \neq PT \subset P'_0$). Thus we have $k + 1 = \text{height} P'' > n + 1$.

As we previously saw $k < n$ we get $n = k$ and height $P'' = n + 1$. Finally the altitude formula applied to $R \subset T$ and $P$ and $P''$ gives height $P + 1 = \text{height} P'' + 0$ so that height $P = n = l(I)$.

Corollary 4. Let $R$ be a Noetherian domain satisfying the altitude formula. Let $0 \neq I \subseteq P$ be ideals of $R$ with $P$ prime. If $P \in \hat{A}(n)$ for any $n > 1$, then height $P = l(IR_P)$.

Proof. By [9, Theorem 2.5], $\hat{A}(n) \subseteq \hat{A}^*$, and so the corollary is immediate from Theorem 3.

We can strengthen Corollary 2 in the case that $I$ is basic. Recall that an ideal in a local domain is basic if $v(I) = l(I)$, or equivalently if $I$ is a minimal reduction of itself. (Notice that in discussing $\hat{A}^*$, one may always assume that $I$ is basic, since if $J$ is a minimal reduction of $I$, making $J$ basic, then for all $n > 1$, $J^n$ reduces $I^n$ so that $\hat{j}^n = \hat{i}^n$.)

Corollary 5. Let $I$ be a basic ideal in a local domain $(R, P)$ which satisfies the altitude formula. Then $P \in \hat{A}^*$ if and only if $PT$ is a height 1 prime of $T$.

Proof. Assume that $P \in \hat{A}^*$. We refer to the second half of the proof of Theorem 3. We have height$(P''/P') = k = n = l(I) = v(I)$. If $I = (a_1, \ldots, a_n)$ then $T = R[ax_1, \ldots, ax_n]$ and we have an obvious homomorphism from $R[x_1, \ldots, x_n]$ onto $T$. Let $Q''$ and $Q'$ be the inverse images of $P''$ and $P'$ respectively. Since $P'' \cap R = P = P' \cap R$, we have $Q'' \cap R = P = Q' \cap R$. Thus $Q''$ and $Q'$ are two primes in $R[x_1, \ldots, x_n]$ both lying over $P$. However height$(Q''/Q') = \text{height}(P''/P') = n$. This forces $Q'$ to be $PR[x_1, \ldots, x_n]$ and so its image $P'$ is $PT$. Thus $PT$ is a height 1 prime of $T$. The converse follows from Corollary 2.
Theorem 6. Let $R$ be a 2-dimensional normal Noetherian domain. Then for any ideal $I$ of $R$, $A^* = \hat{A}^*$. 

Proof. By [9, Corollary 2.6.1], $\hat{A}^* \subseteq A^*$. Conversely, suppose that $P \in A^*$. If $P$ is minimal over $I$ then obviously $P \in \hat{A}^*$. If $P$ is not minimal over $I$ then height $P = 2$. By [5, Proposition 21], $IR_P$ is not principal. I claim that $l(IR_P) > 1$. If $l(IR_P) = 1$ then by the usual method, we may assume $R_P/\mathfrak{p}$ is infinite, so for some $a \in R_P$, $aR_P$ reduces $IR_P$. Thus $aR_P \subseteq IR_P \subseteq aR_P$. However, since $R_P$ is normal, $aR_P = aR_P$, showing that $IR_P$ is principal. This contradiction shows that $l(IR_P) > 1$. We know $l(IR_P) < \text{height } P = 2$. Thus $l(IR_P) = 2 = \text{height } P$. By Theorem 3, $P \in \hat{A}^*$, since 2-dimensional normal Noetherian domains are Cohen-Macaulay and hence satisfy the altitude formula [6, 35.5].

Corollary 7. Let $I$ be an ideal in a 2-dimensional normal Noetherian domain $R$. Let $P$ be prime in $R$ containing $I$. Then $P \in A^*$ if and only if height $P = l(IR_P)$.

Proof. Immediate from Theorems 3 and 6.

Corollary 7 fails without normality. If $R$ is not normal then there will always be elements $a \in R$ for which $(a) \not\subseteq (\hat{a})$. In our next proposition we use this to find an $I$ for which $A^* \neq \hat{A}^*$.

Proposition 8. Let $(R, \mathfrak{p})$ be a local domain with dim $R > 1$. Let $0 \neq a \in R$ and suppose that $y \in (\hat{a}) - (a)$. Let $I = (Py, a)$. Then for all $n > 1$, $P \in A(n)$. If $(R, \mathfrak{p})$ satisfies the altitude formula, then for all $n > 1$, $P \in A(n) - \hat{A}(n)$.

Proof. As $y \in (\hat{a})$, $y$ satisfies an equation $y^m + r_1 ay^{m-1} + \cdots + r_m a^m = 0$. Note $m > 1$ since $y \notin (a)$. Suppose that here $m$ is the least possible. If $1 < n < m$, I claim that $y^n \notin I^n$ for if $y^n \in I^n = (Py, a)^n$, write $y^n = r_0 y^n p_{n-1}^{y-1} p_{n-1} a + \cdots + r_0 p_0 a^n$ with $p_i \in P_i$. Thus $y^n(1 - r_0 p_n) - r_1 a p_{n-1} y^{n-1} - \cdots - r_m p_0 a^n = 0$. This is impossible since $1 - r_0 p_n$ is a unit and $n < m$. Thus $y^n \notin I^n$ for $1 < n < m$. Now $Py^n \subseteq (Py, a)^n = I^n$, and so $P^n$ consists of zero divisors modulo $I^n$. Thus $P \in A(n)$ for $1 < n < m$.

I now claim that $aI^{m-1} = I^m$. Obviously $I^n = (Py)^n + a(Py)^{n-1} + \cdots + a^{n-1} Py + (a)^n$, and each term of this sum is contained in $aI^{m-1}$ except the term $(Py)^n$. However we have $y^n + r_1 a y^{m-1} + \cdots + r_m a^m = 0$ from which we see that $(Py)^n \subseteq aI^{m-1}$. Thus $I^n \subseteq aI^{m-1}$. The other inclusion holds since $a \in I$.

Now consider $n > m$. By the first paragraph of this proof, we already have $P = (I^{m-1}; c)$ for some $c \in R$. Obviously $P = (I^{m-1}; a^n - m + 1; ca^{m-1})$. However since $aI^{m-1} = I^m, a^n - m + 1 I^{m-1} = I^n$, and so $P \in A(n)$ for all $n > 1$.

Finally suppose that $R$ satisfies the altitude formula. Since height $P = \dim R > 1$, in order to show that $P \notin \hat{A}(n)$ for all $n > 1$, in view of Corollary 4, it is enough to show that $l(I) = 1 < \text{height } P$. However the second paragraph of the proof shows that $(a)$ reduces $I$. Thus $l(I) = 1$ as desired.
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