Trace and the Regular Ring of a Finite AW*-Algebra

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Abstract. A finite AW*-algebra is of type I if and only if its maximal ring of quotients has a center-valued trace. In particular, a center-valued trace need not be extendible to the maximal (or classical) ring of quotients.

Let $R$ be a ring with involution $x \mapsto x^*$ (that is, $x^{**} = x$, $(x + y)^* = x^* + y^*$ and $(xy)^* = y^*x^*$ for all $x, y$ in $R$), and let $Z$ be the center of $R$; we say that $R$ has a center-valued trace if there exists a mapping $R \rightarrow Z$, denoted $x \mapsto x^*$, such that (i) $(x + y)^* = x^* + y^*$ for all $x, y$ in $R$, (ii) $(xy)^* = (yx)^*$ for all $x, y$ in $R$, (iii) $z^* = z$ for all $z \in Z$, and (iv) for each $x \in R$, $(x^*x)^*\xi$ is a finite sum of elements of the form $z^*z$ with $z \in Z$ (a "positivity" condition).

Let $A$ be a finite AW*-algebra and let $C$ be its regular ring as constructed, for example, in [4, Chapter 8]. (Thus $C$ is a regular Baer $*$-ring [4, Theorem 1, p. 220 and Theorem 1, p. 235]. We remark that, by a theorem of J. E. Roos [10], $C$ may be identified with the maximal ring of quotients (right or left) of $A$ (cf. [7], [9]). Moreover, for every $x \in C$ one can write $x = ab^{-1}$ with $a = x(1 + x^*x)^{-1}$ and $b = (1 + x^*x)^{-1}$, and one has $a^*a < 1$, $0 < b < 1$ (cf. [4, Exercise 3, p. 242]); it follows that $C$ is a classical ring of quotients of $A$ [8, p. 108], a remark valid with $A$ replaced by any finite Baer $*$-ring satisfying the axioms $1^* = 6^*$ of [4, pp. 248–249]. In the foregoing remarks, statements about right quotients and left quotients are equally valid, since the involution of $A$ extends to $C$.)

It is not known if, in general, $A$ possesses a center-valued trace; it is known that a trace exists if $A$ is of type I (cf. [2, proof of Theorem 5, p. 178]) or if $A$ is a (finite) von Neumann algebra [6, Theorem 1, p. 288]. In the following theorem, the question of existence of trace for $A$ is not begged; the theorem shows that even if $A$ possesses a trace, $C$ need not.

Theorem 1. Let $A$ be a finite AW*-algebra, $C$ its regular ring. In order that $C$ admit a center-valued trace, it is necessary and sufficient that $A$ be of type I.

Proof. The proof that $C$ possesses a center-valued trace when $A$ is of type I is given in [2, Theorem 5]. In general, $A$ is the sum of a type I algebra and a type II algebra (cf. [4, Theorem 2, p. 94]); assuming $A$ to be of type II, the proof will be completed by showing that $C$ does not possess a center-valued trace. Let $(g_n)$ be a sequence of pairwise orthogonal projections in $A$ such that $\sup g_n = 1$ and $D(g_n) = 2^{-n}1$ for all $n (n = 1, 2, 3, \cdots)$, $D$ being the center-valued dimension function.
of \( A \) [cf. 4, Theorem 1, p. 181 and Proposition 15, p. 159]. Write \( x_n = \sum_{k=1}^{n} 2^{k}g_k \), \( e_n = \sum_{k=1}^{n} g_k \); if \( m < n \) then \( x_n e_m = x_m = e_m x_n \), thus there exists \( x \in C \) such that \( x e_n = x_n \) for all \( n \) [4, Proposition 1, p. 219]. Then [4, Proposition 6, p. 242] one has \( x > 0 \) and \( x_n = x^{1/2}e_n x^{1/2} < x^{1/2}x^{1/2} = x \), thus \( 0 < x_n < x \) for all \( n \). Assume to the contrary that \( C \) admits a center-valued trace \( \tau \). It follows from uniqueness of dimension that \( e^b = D(e) \) for all projections \( e \) [4, Theorem 1, p. 181]; therefore \( (x_n)^b = \sum_{k=1}^{n} 2^k D(g_k) = n1 \), thus \( 0 < n1 = (x_n)^b < x^b \) for all \( n \). Then \( 0 < (x^b)^{-1} < (n1)^{-1} \) for all \( n \) [3, Proposition 8.12], thus \( (x^b)^{-1} \in A \) and \( ||(x^b)^{-1}|| < 1/n \) for all \( n \), which yields the absurdity \( (x^b)^{-1} = 0 \). □

It is curious that although \( C \) does not in general possess a trace, it exhibits the following trace-like behavior: the equation \( x^*x = xx^* = 1 \) has no solution in \( C \) [5, Lemma, p. 619]. Also, Theorem 1 thwarts any prospect of proving a theorem of Fuglede type in \( C \) by means of a trace argument (cf. [2, Theorems 4 and 5], [5, Theorem 6]).

For \( A \) a finite Baer *-ring, it is not clear what “trace” should mean. The dimension function \( D \) takes its values in the space of continuous complex-valued functions \( C(\mathcal{X}) \), where \( \mathcal{X} \) is the Stone representation space (i.e., the spectrum) of the complete Boolean algebra of central projections of \( A \) [4, p. 153]. Since not every element of \( A \) need be bounded in the sense of [4, Definition 1, p. 243], and since a trace function should in some sense extend the dimension function, a good candidate for the value-space of a trace is the regular ring \( \hat{C}(\mathcal{X}) \) of the commutative AW*-algebra \( \hat{C}(\mathcal{X}) \). Let us say that \( A \) has a spectral trace if there exists a mapping \( A \rightarrow \hat{C}(\mathcal{X}) \), denoted \( x \mapsto x^b \), such that \( i \) \( (x + y)^b = x^b + y^b \), \( (ii) \ (xy)^b = (yx)^b \), \( (iii) \ h^b = h \) for all central projections \( h \), and \( (iv) \ (x^*x)^b > 0 \) for all \( x \in A \).

When \( A \) is a finite AW*-algebra, the concept of spectral trace coincides with that of center-valued trace defined earlier; for, in this case, the center \( Z \) of \( A \) may be identified with \( C(\mathcal{X}) \), and the center of \( C \) with \( \hat{C}(\mathcal{X}) \) [1, Theorem 9.2]. For \( A \) a finite Baer *-ring of type I, the construction of trace in Theorem 1 breaks down (basically because an abelian ring need not be commutative), and, as the following theorem shows, the bad news persists for rings of type II (so to speak, \( \hat{C}(\mathcal{X}) \) is no better a value space for trace, than is the center of \( C \)).

**Theorem 2.** If \( A \) is a finite Baer *-ring of type II, satisfying the axioms 1°–5° of [4, p. 248], then the regular ring \( C \) of \( A \) does not admit a spectral trace.

**Proof.** The proof proceeds as in Theorem 1, up through the point that \( 0 < (x^b)^{-1} < (n1)^{-1} \) for all \( n \). In particular, \( 0 < (x^b)^{-1} < 1 \), therefore there exists a projection \( e \) such that \( D(e) = (x^b)^{-1} \) [4, Theorem 3, p. 182]. Let \( f \) be a simple projection such that \( f < e \) [4, Proposition 16, p. 159], let \( h \) be the central cover of \( f \), and let \( r \) be the integer such that \( D(f) = (1/r)h \); then \( 0 < (1/r)h = D(f) < D(e) = (x^b)^{-1} < (n1)^{-1} \) for all \( n \), and for \( n = 2r \) this yields \( 2h < 1 \), \( h < 1 - h \), whence \( h = 0 \), a contradiction. □

**References**


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