TWO NEW EXTREMAL PROPERTIES
OF THE KOEBE-FUNCTION\(^{1}\)

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Abstract. Using essentially Löwner's method the extremality of the Koebe-functions with respect to two coefficient problems for inverses of univalent functions is proved.

Let \( D = \{ z \mid |z| < 1 \} \) and \( S = \{ f \mid f \) regular and univalent in \( D, f(0) = f'(0) - 1 = 0 \} \). K. Löwner \([4]\) proved: If \( F(w) = w + \sum_{n=2}^{\infty} A_n w^n \) is the inverse of a function in \( S \), then

\[ |A_n| < \frac{(2n)!}{n! (n + 1)!} \]

with equality only for the inverses of the Koebe-functions \( k_\sigma(z) = z(1 + \sigma z)^{-2}, |\sigma| = 1 \).

In this note we shall prove similar results for the functions

\[ \ln F'(w), \quad \Delta(F(w), w) := \left( \frac{F''}{F'} \right)' - \frac{1}{2} \left( \frac{F''}{F'} \right)^2. \]

This work was stimulated by a conjecture of the first author (see \([2]\) and \([3]\)) and the preprint \([6]\) of a lecture given by G. Schober at the Durham Conference on Aspects of Contemporary Complex Analysis in 1979.

Theorem. Let \( F \) be the inverse of a function in \( S, K_1(w) = k_1^{-1}(w), \)

\[
\ln F'(w) = \sum_{n=1}^{\infty} B_n w^n, \quad \ln K_1(w) = \sum_{n=1}^{\infty} b_n w^n,
\]

\[
\Delta(F(w), w) = \sum_{n=0}^{\infty} C_n w^n, \quad \Delta(K_1(w), w) = \sum_{n=0}^{\infty} c_n w^n.
\]

Then \( |B_n| < b_n \) for \( n \in \mathbb{N} \) and \( |C_n| < c_n \) for \( n \in \mathbb{N} \cup \{0\} \). Equality for \( n \in \mathbb{N} \) occurs only for the functions \( K_\sigma(w) = k_\sigma^{-1}(w), |\sigma| = 1 \).

Remarks. In the case of the Schwarzian derivative \( \Delta(K_1(w), w) \) we have the simple representation \( c_n = 4^n 6(n + 1), n \in \mathbb{N} \cup \{0\} \) (see \([3]\)). The first part of the theorem implies Löwner's theorem since each \( A_n \) is a polynomial with positive coefficients in the \( B_n \).\(^{2}\)

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Proof. The proof follows the same line as the famous proof of Löwner’s result (see f. i. [1], [4], [6]). So we need only give here the crucial steps.  

If \( f \in S \), \( f \) can be embedded into a subordination chain. It results that \( F \), the inverse of \( f \), has a representation

\[
F(w) = \lim_{t \to \infty} \Phi(e^{-w}, t), \quad \partial \Phi(w, t)/\partial t = w(\partial \Phi(w, t)/\partial w)p(w, t) \tag{1}
\]

with

\[
p(w, t) = 1 + \sum_{n=1}^{\infty} p_n(t)w^n, \quad \text{Re} \ p(w, t) > 0 \quad \text{for} \ w \in D, \ t > 0, \ \Phi(w, 0) = w. \tag{2}
\]

(For details see [5].)

Using (1) and (2) and setting

\[
L(w, t) := \ln \frac{\partial \Phi(w, t)}{\partial w} = \sum_{n=0}^{\infty} B_n(t)w^n;
\]

\[
\Delta(w, t) := \Delta(\Phi(w, t), w) = \sum_{n=0}^{\infty} C_n(t)w^n;
\]

we get

\[
\partial L/\partial t = (\partial L/\partial w)wp + (\partial /\partial w)(wp),
\]

\[
\partial \Delta/\partial t = (\partial \Delta/\partial w)wp + 2\Delta(\partial /\partial w)(wp) + (\partial^3 /\partial w^3)(wp),
\]

\[
B_0(t) = t, \quad B_n(t) = \int_0^t e^{n(t-\tau)} \left( \sum_{j=1}^{n-1} jB_j(\tau)p_{n-j}(\tau) + (n+1)p_n(\tau) \right) d\tau, \quad n \in \mathbb{N}, \tag{3}
\]

\[
C_n(t) = \int_0^t e^{(n+2)(t-\tau)} \left( \sum_{j=0}^{n-1} C_j(\tau)p_{n-j}(\tau)(2n-j+2) + \frac{(n+3)!}{n!}p_{n+2}(\tau) \right) d\tau,
\]

\[n \in \mathbb{N} \cup \{0\}, \tag{4}
\]

\[
B_n = \lim_{t \to \infty} e^{-n}B_n(t), \quad n \in \mathbb{N}, \tag{5}
\]

\[
C_n = \lim_{t \to \infty} e^{-(n+2)/2}C_n(t), \quad n \in \mathbb{N} \cup \{0\}. \tag{6}
\]

(3) and (4) show that \( \text{Re} \ B_n(t) \), resp. \( \text{Re} \ C_n(t) \) is maximal for fixed \( t \) if and only if we choose \( B_j(\tau), \ j = 1, \ldots, n-1 \), resp. \( C_j(\tau), \ j = 0, \ldots, n-1, \ \tau \in [0, t] \) real and maximal and any \( p_j(\tau) \) involved in (3), resp. (4), equal to the constant 2. As a consequence of (5) and (6) we get that Max \( \text{Re} \ B_n \), resp. Max \( \text{Re} \ C_n \), \( n \in \mathbb{N} \), is attained if and only if \( p_1(t) \equiv 2 \) which means \( p(w, t) = (1 + w)/(1 - w) \). Now the assertion of the theorem for \( n \in \mathbb{N} \) follows from the fact that the problems of finding the maximum of the real part and the maximum of the modulus for the given coefficients are equivalent (up to a rotation).

The equality \( C_0 = -f^{(3)}(0) + \frac{3}{2}(f''(0))^2 \) shows that the remaining case is a classical inequality.
REFERENCES


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