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FINITE GROUPS AND INVARIANT SOLUTIONS TO ONE-DIMENSIONAL PLATEAU PROBLEMS

DAVID BINDSCHADLER

ABSTRACT. Let G be a finite group of isometries acting on a complete Riemannian manifold. Suppose that B is a 0-dimensional boundary which is G-invariant. If the order of G divides the product of the cardinality of the orbit and the density of B at each point, then a G-invariant absolutely length minimizing integral current with boundary B can be constructed.

1. Introduction. Let G be a finite group of isometries acting on a complete Riemannian manifold. Let B denote a boundary consisting of a finite collection of points with integral densities. It is shown that if the order of the group divides the product of the cardinality of the orbit and the density of B at each point, then there exists an invariant solution to the oriented Plateau problem. (That is, there exists an invariant collection of oriented length minimizing geodesics with boundary B which minimizes total length among all collections of oriented curves with boundary B.) Known examples show that if the divisibility condition fails, there may exist boundaries with no invariant solution. (See [**F1**, 5.4.17].)

For other results concerning invariant solutions to the Plateau problem see [**L**], [**B**] and [**BJ**].

The author would like to thank the referee for suggesting the use of polyhedral chains instead of more abstract objects.

2. Polyhedral 1-chains in a metric space.

2.1. *Definitions and notations*. Let (X, d) be a metric space.

A 0-*chain* in X is an integer valued function β on X such that $\{x: \beta(x) \neq 0\}$ is a finite set.

A *polyhedral* 1-*chain* (or 1-pchain) in X is a nonnegative function $\alpha: X \times X \to \mathbf{Z}$ such that $\{(x, y) \in X \times X: \alpha(x, y) \neq 0\}$ is finite and $\alpha(x, y)\alpha(y, x) = 0$ for all $(x, y) \in X \times X$.

Let δ_x and $[x, y]$ denote the characteristic functions of $\{x\} \subset X$ and $\{(x, y)\} \subset X \times X$, respectively. Then for any 0-chain β and 1-pchain α, we have

$$\beta = \sum_{x \in X} \beta(x)\delta_x \quad \text{and} \quad \alpha = \sum_{(x,y) \in X \times X} \alpha(x, y)[x, y].$$

If β is a 0-chain and α is a 1-pchain, we define the *mass* of β, the *mass* of α and

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the *boundary* of α by the equations

$$\mathbf{M}(\beta) = \sum_{x \in X} |\beta(x)|,$$

$$\mathbf{M}(\alpha) = \sum_{(x,y) \in X \times X} \alpha(x, y)\, d(x, y),$$

$$\partial\alpha = \sum_{(x,y) \in X \times X} \alpha(x, y)(\delta_y - \delta_x) = \sum_{y \in X} \sum_{x \in X} (\alpha(x, y) - \alpha(y, x))\delta_y.$$

A *cycle* is a 1-pchain with zero boundary.

If α and γ are two 1-pchains their *chain sum* is given by

$$\alpha \oplus \gamma(x, y) = \max\{0, \alpha(x, y) - \alpha(y, x) + \gamma(x, y) - \gamma(y, x)\}$$

for all $(x, y) \in X \times X$. The set of 1-pchains forms a commutative group under chain sums. Indeed, closure, existence of identity and commutativity are obvious. The inverse of α is the 1-pchain α given by $\ominus\alpha(x, y) = \alpha(y, x)$. And associativity follows from the equality $a = \max\{0, a\} - \max\{0, -a\}$. Also observe that

$$\alpha \oplus \gamma(x, y) + \alpha \oplus \gamma(y, x) = |\alpha(x, y) - \alpha(y, x) + \gamma(x, y) - \gamma(y, x)|,$$
$$\partial \ominus \alpha = -\partial\alpha \quad \text{and} \quad \partial(\alpha \oplus \gamma) = \partial\alpha \oplus \partial\gamma.$$

A 1-pchain is called *simple* if it has the form $\Sigma_{i=1}^{l}[x_{i-1}, x_i]$, where $x_i = x_j$ implies $i = j$ or $\{i, j\} = \{0, l\}$.

In case X were a complete Riemannian manifold and $T \in \mathbf{I}_1(X)$ (see [**F1**, pp. 670, 671]) consisted of a sum of oriented length minimizing geodesic arcs with integral density, then one can associate with T a 1-pchain α such that $\mathbf{M}(T) = \mathbf{M}(\alpha)$ and $\partial\alpha$ is essentially ∂T. Also, given a 1-pchain α, we can associate a sum of oriented length minimizing geodesic arcs with integral densities T by associating with $\alpha(x, y) > 0$ any length minimizing geodesic arc with boundary $\boldsymbol{\delta}_y - \boldsymbol{\delta}_x$ with density $\alpha(x, y)$. Then T and α have the same boundary and $\mathbf{M}(T) \leq \mathbf{M}(\alpha)$. It is this connection that lets us exploit mass minimizing 1-pchain in the study of mass minimizing integral currents.

2.2. THEOREM. *Given a polyhedral* 1-*chain* α, *there exists simple* 1-*pchains* σ_i, $i = 1, \ldots, l$, *such that* $\alpha = \ominus_{i=1}^{l} \sigma_i = \Sigma_{i=1}^{l} \sigma_i$, $\mathbf{M}(\alpha) = \Sigma_{i=1}^{l} \mathbf{M}(\sigma_i)$, *and* $\mathbf{M}(\partial\alpha) = \Sigma_{i=1}^{l} \mathbf{M}(\partial\sigma_i)$.

PROOF. By induction on the integer $\Sigma_{(x,y) \in X \times X} \alpha(x, y)$, it suffices to find a nonzero simple 1-pchain σ so that

(i) $\alpha \ominus \sigma = \alpha - \sigma$,

(ii) $\mathbf{M}(\alpha) = \mathbf{M}(\alpha \ominus \sigma) + \mathbf{M}(\sigma)$,

(iii) $\mathbf{M}(\partial\alpha) = \mathbf{M}(\partial(\alpha \ominus \sigma)) + \mathbf{M}(\partial\sigma)$.

This will be accomplished by finding points $x_M, \ldots, x_N$, $M, N \in \mathbf{Z}$ with $M < N$ such that

(a) $x_i \neq x_j$ unless $i = j$ or $\{i, j\} = \{M, N\}$,

(b) $\alpha(x_{i-1}, x_i) > 0$ for $i = M + 1, \ldots, N$,

(c) either (1) $x_M = x_N$ or (2) $\partial\alpha(x_M) < 0 < \partial\alpha(x_N)$.

For setting $\sigma = \sum_{i=M+1}^{N}[x_{i-1}, x_i]$, one obtains (i), hence (ii), from (a) and (b) and (iii) from (c).

The theorem is trivial if $\alpha = 0$, so choose x_0 and x_1 such that $\alpha(x_0, x_1) > 0$. If $\partial\alpha(x_1) > 0$, then set $N = 1$. If $\partial\alpha(x_1) \leqslant 0$, then we may assume that distinct points $x_0, \ldots, x_k$ have been chosen such that $\alpha(x_{i-1}, x_i) > 0 \geqslant \partial\alpha(x_i)$ for all $i \in \{1, \ldots, k\}$. Since $0 \geqslant \partial\alpha(x_k) = \sum_{y \in X} \alpha(y, x_k) - \alpha(x_k, y)$, there exists x_{k+1} with the property that $\alpha(x_k, x_{k+1}) > 0$.

If $x_{k+1} = x_i$ for some $i \in \{0, \ldots, k\}$, then $x_i, \ldots, x_{k+1}$ satisfies (a), (b), (c)(1).

If $x_{k+1} \notin \{x_0, \ldots, x_k\}$ and $\partial\alpha(x_{k+1}) > 0$, then set $N = k + 1$.

If $x_{k+1} \notin \{x_0, \ldots, x_k\}$ and $\partial\alpha(x_{k+1}) \leqslant 0$, then continue.

Since $\{(x, y)\colon \alpha(x, y) > 0\}$ is a finite set, we either obtain the desired set or distinct points $x_0, \ldots, x_N$ such that $\alpha(x_N) > 0$ and $\alpha(x_{i-1}, x_i) > 0$ for all $i \in \{1, \ldots, N\}$. In the latter case a similar argument using decreasing indices can be used to complete the proof.

2.3. Given a 0-chain β such that $\sum_{x \in X} \beta(x) = 0$, we denote by $\mathbf{M}_\beta$ the number

$$\inf\{\mathbf{M}(\alpha)\colon \alpha \text{ is a 1-pchain with } \partial\alpha = \beta\}.$$

Observe that $\mathbf{M}_\beta$ is finite since one can construct a 1-pchain with boundary β by adding an appropriate number of characteristic functions $[x, y]$ where $\beta(x) < 0 < \beta(y)$. Note also that $\mathbf{M}_\beta = \inf\{\mathbf{M}(\alpha_0 \oplus \zeta)\colon \zeta \text{ is a cycle}\}$, where α_0 is any 1-pchain with $\partial\alpha_0 = \beta$.

2.4. THEOREM. *If β is a 0-chain in X such that $\sum \beta(x) = 0$, then there exists a 1-pchain α_0 with $\partial\alpha_0 = \beta$ and $\mathbf{M}(\alpha_0) = \mathbf{M}_\beta$.*

PROOF. For any 1-pchain α with $\partial\alpha = \beta$ one can find σ_i, $i = 1, \ldots, l$, as in 2.2. Now define

$$\gamma_i = \begin{cases} [x, y] & \text{if } \partial\sigma_i = \delta_y - \delta_x, \\ 0 & \text{if } \partial\sigma_i = 0, \end{cases}$$

and $\gamma = \bigoplus_{i=1}^{l} \gamma_i = \sum_{i=1}^{l} \gamma_i$. Then $\partial\gamma = \sum_{i=1}^{l} \partial\sigma_i = \alpha$ and by the triangle inequality

$$\mathbf{M}(\gamma) = \sum_{i=1}^{l} \mathbf{M}(\gamma_i) \leqslant \sum_{i=1}^{l} \mathbf{M}(\sigma_i) = \mathbf{M}(\alpha).$$

Observing that $\gamma(x, y) > 0$ implies $\beta(x) < 0 < \beta(y)$ we conclude

$$\mathbf{M}_\beta = \inf\{\mathbf{M}(\gamma)\colon \partial\gamma = \beta \text{ and } \gamma(x, y) > 0 \text{ implies } \beta(x) < 0 < \beta(y)\}.$$

The infimum is attained, since the latter set is finite.

2.5. THEOREM. *For any positive integer k and 0-chain β in X with $\sum_{x \in X} \beta(x) = 0$, the equality $\mathbf{M}_{k\beta} = k\mathbf{M}_\beta$ holds.*

PROOF. If α is a 1-pchain, then $\partial k\alpha = k\partial\alpha$ and $\mathbf{M}(k\alpha) = k\mathbf{M}(\alpha)$ from which $\mathbf{M}_{k\beta} \leqslant k\mathbf{M}_\beta$ follows.

For the reverse inequality, we suppose α is a 1-pchain such that $\partial\alpha = \beta$ and $\mathbf{M}(\alpha) = \mathbf{M}_\beta$. Given any cycle ζ, we use 2.2 to decompose ζ into a sum of simple cycles σ_i, $i = 1, \ldots, l$. For each $i \in \{1, \ldots, l\}$ and $(x, y) \in X \times X$, let

$$s_i(x,y) = \begin{cases} -1 & \text{if } (\alpha(x,y) - \alpha(y,x))(\sigma_i(x,y) - \sigma_i(y,x)) < 0, \\ 1 & \text{otherwise.} \end{cases}$$

Since

$$\begin{aligned} \alpha \oplus \sigma_i(x,y) + \alpha \oplus \sigma_i(y,x) &= |\alpha(x,y) - \alpha(y,x) + \sigma_i(x,y) - \sigma_i(y,x)| \\ &= |\alpha(x,y) - \alpha(y,x)| + s_i(x,y)|\sigma_i(x,y) - \sigma_i(y,x)|, \end{aligned}$$

we have

$$\mathbf{M}(\alpha \oplus \sigma_i) = \mathbf{M}(\alpha) + \frac{1}{2} \sum_{(x,y)\in X\times X} s_i(x,y)|\sigma_i(x,y) - \sigma_i(y,x)|\, d(x,y).$$

By the mass minimality of α, we conclude

$$\sum_{(x,y)\in X\times X} s_i(x,y)|\sigma_i(x,y) - \sigma_i(y,x)|\, d(x,y) \geq 0.$$

For each $(x,y) \in X \times X$ let $I(x,y) = \{i\colon s_i(x,y) < 0\}$. Then

$$\begin{aligned} k\alpha \oplus \sum_{i=1}^{l} \sigma_i(x,y) + k\alpha \oplus \sum_{i=1}^{l} \sigma_i(y,x) \\ &= \left| k\alpha(x,y) - k\alpha(y,x) + \sum_{i=1}^{l} \sigma_i(x,y) - \sigma_i(y,x) \right| \\ &= \Bigg| k\alpha(x,y) - k\alpha(y,x) + \sum_{i\notin I(x,y)} \sigma_i(x,y) - \sigma_i(y,x) \\ &\qquad + \sum_{i\in I(x,y)} \sigma_i(x,y) - \sigma_i(y,x) \Bigg| \\ &= \Bigg| |k\alpha(x,y) - k\alpha(y,x)| + \sum_{i\notin I(x,y)} |\sigma_i(x,y) - \sigma_i(y,x)| \\ &\qquad - \sum_{i\in I(x,y)} |\sigma_i(x,y) - \sigma_i(y,x)| \Bigg| \\ &= \left| |k\alpha(x,y) - k\alpha(y,x)| + \sum_{i=1}^{l} s_i(x,y)|\sigma_i(x,y) - \sigma_i(y,x)| \right| \\ &\geq |k\alpha(x,y) - k\alpha(y,x)| + \sum_{i=1}^{l} s_i(x,y)|\sigma_i(x,y) - \sigma_i(y,x)|. \end{aligned}$$

Hence

$$\begin{aligned} \mathbf{M}(k\alpha \oplus \zeta) &\geq \mathbf{M}(k\alpha) + \frac{1}{2} \sum_{(x,y)\in X\times X} \sum_{i=1}^{l} s_i(x,y)|\sigma_i(x,y) - \sigma_i(y,x)|\, d(x,y) \\ &\geq \mathbf{M}(k\alpha) = k\mathbf{M}_\beta. \end{aligned}$$

Thus the inequality $\mathbf{M}_{k\beta} \geq k\mathbf{M}_\beta$ is established.

3. Invariant solutions to Plateau problems.

3.1. Let G be a finite group acting on X so that $d(g(x), g(y)) = d(x,y)$ for all x, $y \in X$ and $g \in G$. Let $G(x)$ denote the orbit of x and G_x denote the isotropy

subgroup of G at x. Identifying orbits of G to points gives the orbit space Z. Let π: $X \to Z$ be the canonical projection. For each $u, v \in Z$ define

$$d'(u, v) = \min\{d(x, y): x \in \pi^{-1}(u), y \in \pi^{-1}(v)\}.$$

The 0-chain β and the 1-chain α are called invariant if $\beta(gx) = \beta(x)$ and $\alpha(g(x), g(y)) = \alpha(x, y)$ for all $x, y \in X$.

3.2. THEOREM. *If β is a G-invariant 0-chain with $\sum_{x \in X} \beta(x) = 0$ and the order of G divides $\beta(x)$ card $G(x)$ for each $x \in X$, then there exists a G-invariant 1-pchain α such that $\partial\alpha = \beta$ and $\mathbf{M}(\alpha) = \mathbf{M}_\beta$.*

PROOF. Let r be the order of G. Define β': $Z \to \mathbf{Z}$ by

$$\beta'(u) = \frac{1}{r} \sum_{x \in \pi^{-1}(u)} \beta(x).$$

Since β is G-invariant and r divides $\beta(x)$ card $G(x)$, β' is a 0-chain in Z. If γ is a 1-pchain in X with $\partial\gamma = \beta$, then one can define the 1-pchain in Z by

$$\gamma'(u, v) = \sum_{(x,y) \in \pi^{-1}(u) \times \pi^{-1}(v)} \gamma(x, y).$$

Then $\partial\gamma' = r\beta'$ and $\mathbf{M}(\gamma') \leq \mathbf{M}(\gamma)$. Hence by 2.5, $\mathbf{M}_\beta \geq \mathbf{M}_{r\beta'} = r\mathbf{M}_{\beta'}$.

Now choose a 1-pchain α' such that $\partial\alpha' = \beta'$ and $\mathbf{M}(\alpha') = \mathbf{M}_{\beta'}$. For each $(u, v) \in Z \times Z$ such that $\alpha'(u, v) > 0$, we find $x_u \in \pi^{-1}(u)$ and $y_{u,v} \in \pi^{-1}(v)$ such that $d(x_u, y_{u,v}) = d'(u, v)$. Let α be the 1-pchain in X given by

$$\alpha = \sum_{(u,v) \in Z \times Z} \sum_{g \in G} \alpha'(u, v)[g(x_u), g(y_{u,v})].$$

Then

$$\begin{aligned}\mathbf{M}(\alpha) &= \sum_{(u,v) \in Z \times Z} \sum_{g \in G} \alpha(g(x_u), g(y_{u,v}))\, d(x_u, y_{u,v}) \\ &= \sum_{(u,v) \in Z \times Z} \sum_{g \in G} \alpha'(u, v)\, d'(u, v) = r\mathbf{M}(\alpha'),\end{aligned}$$

and

$$\begin{aligned}\partial\alpha &= \sum_{(u,v) \in Z \times Z} \sum_{g \in G} \alpha'(u, v)\delta_{gy_{u,v}} - \sum_{(u,v) \in Z \times Z} \sum_{g \in G} \alpha'(u, v)\delta_{gx_u} \\ &= \sum_{u \in Z} \left(\sum_{v \in Z} \alpha'(v, u) - \alpha'(u, v) \right) \sum_{g \in G} \delta_{gw_u} \\ &= \sum_{u \in Z} \beta'(u) \sum_{g \in G} \delta_{gw_u},\end{aligned}$$

where w_u is any point in $\pi^{-1}(u)$. Since (card $G(x)$)(card G_x) $= r$, we have that

$$\partial\alpha(x) = \beta'(\pi(x)) \text{ card } G_x = \beta(x).$$

Hence $\mathbf{M}_\beta \leq \mathbf{M}(\alpha) = r\mathbf{M}(\alpha') = r\mathbf{M}_{\beta'} \leq \mathbf{M}_\beta$.

3.3. For an explanation of the terms and notation used in the following corollary, we refer the reader to [**F1**, pp. 670, 671].

COROLLARY. *Let X be a complete Riemannian manifold and G be a finite group of isometries of X. If B is a G-invariant 0-dimensional rectifiable current which is a boundary and r divides $\Theta(\|B\|, x)$ card $G(x)$ for all $x \in X$, then there exists a G-invariant $T \in \mathbf{I}_1(X)$ such that $\partial T = B$ and*

$$\mathbf{M}(T) = \inf\{\mathbf{M}(R) \colon R \in \mathbf{I}_1(X) \text{ and } \partial R = B\}.$$

PROOF. Repeat the argument of 3.2 except in the definition of α use $g_{\#}L(x_u, y_{u,v})$ where $L(x_u, y_{u,v})$ is any oriented length minimizing arc from x_u to $y_{u,v}$.

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DEPARTMENT OF MATHEMATICS, WAYNE STATE UNIVERSITY, DETROIT, MICHIGAN 48202