FINITE GROUPS AND INVARIANT SOLUTIONS TO ONE-DIMENSIONAL PLATEAU PROBLEMS

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Abstract. Let $G$ be a finite group of isometries acting on a complete Riemannian manifold. Suppose that $B$ is a 0-dimensional boundary which is $G$-invariant. If the order of $G$ divides the product of the cardinality of the orbit and the density of $B$ at each point, then a $G$-invariant absolutely length minimizing integral current with boundary $B$ can be constructed.

1. Introduction. Let $G$ be a finite group of isometries acting on a complete Riemannian manifold. Let $B$ denote a boundary consisting of a finite collection of points with integral densities. It is shown that if the order of the group divides the product of the cardinality of the orbit and the density of $B$ at each point, then there exists an invariant solution to the oriented Plateau problem. (That is, there exists an invariant collection of oriented length minimizing geodesics with boundary $B$ which minimizes total length among all collections of oriented curves with boundary $B$.) Known examples show that if the divisibility condition fails, there may exist boundaries with no invariant solution. (See [F1, 5.4.17].)

For other results concerning invariant solutions to the Plateau problem see [L], [B] and [BJ].

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2. Polyhedral 1-chains in a metric space.

2.1. Definitions and notations. Let $(X,d)$ be a metric space.

A 0-chain in $X$ is an integer valued function $\beta$ on $X$ such that $\{x: \beta(x) \neq 0\}$ is a finite set.

A polyhedral 1-chain (or 1-pchain) in $X$ is a nonnegative function $\alpha: X \times X \to \mathbb{Z}$ such that $\{(x,y) \in X \times X: \alpha(x,y) \neq 0\}$ is finite and $\alpha(x,y)\alpha(y,x) = 0$ for all $(x,y) \in X \times X$.

Let $\delta_x$ and $[x,y]$ denote the characteristic functions of $\{x\} \subset X$ and $\{(x,y)\} \subset X \times X$, respectively. Then for any 0-chain $\beta$ and 1-pchain $\alpha$, we have

$$\beta = \sum_{x \in X} \beta(x) \delta_x \quad \text{and} \quad \alpha = \sum_{(x,y) \in X \times X} \alpha(x,y) [x,y].$$

If $\beta$ is a 0-chain and $\alpha$ is a 1-pchain, we define the mass of $\beta$, the mass of $\alpha$ and
the boundary of $\alpha$ by the equations

$$M(\beta) = \sum_{x \in X} |\beta(x)|,$$

$$M(\alpha) = \sum_{(x,y) \in X \times X} \alpha(x,y) d(x,y),$$

$$\partial \alpha = \sum_{(x,y) \in X \times X} \alpha(x,y)(\delta_y - \delta_x) = \sum_{y \in X} \sum_{x \in X} (\alpha(x,y) - \alpha(y,x)) \delta_y.$$

A cycle is a 1-pchain with zero boundary.

If $\alpha$ and $\gamma$ are two 1-pchains their chain sum is given by

$$\alpha \oplus \gamma(x,y) = \max\{0, \alpha(x,y) - \alpha(y,x) + \gamma(x,y) - \gamma(y,x)\}$$

for all $(x,y) \in X \times X$. The set of 1-pchains forms a commutative group under chain sums. Indeed, closure, existence of identity and commutativity are obvious. The inverse of $\alpha$ is the 1-pchain $\alpha$ given by $\Theta \alpha(x,y) = \alpha(y,x)$. And associativity follows from the equality $\alpha = \max(0, \alpha) - \max(0, -\alpha)$. Also observe that

$$\alpha \oplus \gamma(x,y) = \alpha \oplus \gamma(y,x) = |\alpha(x,y) - \alpha(y,x) + \gamma(x,y) - \gamma(y,x)|,$$

$$\partial \Theta \alpha = -\partial \alpha \quad \text{and} \quad \partial(\alpha \oplus \gamma) = \partial(\alpha \oplus \gamma).$$

A 1-pchain is called simple if it has the form $\Sigma_{i=1}^l [x_{i-1}, x_i]$, where $x_i = x_j$ implies $i = j$ or $(i, j) = (0, l)$.

In case $X$ were a complete Riemannian manifold and $T \in I_1(X)$ (see [F1, pp. 670, 671]) consisted of a sum of oriented length minimizing geodesic arcs with integral density, then one can associate with $T$ a 1-pchain $\alpha$ such that $M(T) = M(\alpha)$ and $\partial \alpha$ is essentially $\partial T$. Also, given a 1-pchain $\alpha$, we can associate a sum of oriented length minimizing geodesic arcs with integral densities $T$ by associating with $\alpha(x,y) > 0$ any length minimizing geodesic arc with boundary $\delta_y - \delta_x$ with density $\alpha(x,y)$. Then $T$ and $\alpha$ have the same boundary and $M(T) < M(\alpha)$. It is this connection that lets us exploit mass minimizing 1-pchain in the study of mass minimizing integral currents.

2.2. Theorem. Given a polyhedral 1-chain $\alpha$, there exists simple 1-pchains $\sigma_i$, $i = 1, \ldots, l$, such that $\alpha = \bigoplus_{i=1}^l \sigma_i = \Sigma_{i=1}^l \sigma_i$, $M(\alpha) = \Sigma_{i=1}^l M(\sigma_i)$, and $M(\partial \alpha) = \Sigma_{i=1}^l M(\partial \sigma_i)$.

Proof. By induction on the integer $\Sigma_{(x,y) \in X \times X} \alpha(x,y)$, it suffices to find a nonzero simple 1-pchain $\sigma$ so that

(i) $\alpha \oplus \sigma = \alpha - \sigma$,

(ii) $M(\alpha) = M(\alpha \oplus \sigma) + M(\sigma)$,

(iii) $M(\partial \alpha) = M(\partial(\alpha \oplus \sigma)) + M(\partial \sigma)$.

This will be accomplished by finding points $x_M, \ldots, x_N, M, N \in Z$ with $M < N$ such that

(a) $x_i \neq x_j$ unless $i = j$ or $(i, j) = (M, N),$

(b) $\alpha(x_{i-1}, x_i) > 0$ for $i = M + 1, \ldots, N$,

(c) either (1) $x_M = x_N$ or (2) $\partial \alpha(x_M) < 0 < \partial \alpha(x_N).$
For setting \( o = 2f_M + \), one obtains (i), hence (ii), from (a) and (b) and (iii) from (c).

The theorem is trivial if \( a = 0 \), so choose \( x_0 \) and \( x_\alpha \) such that \( a(x_0, x_\alpha) > 0 \). If \( 3a(x_0) > 0 \), then set \( N = 1 \). If \( 3a(x_0) < 0 \), then we may assume that distinct points \( x_0, \ldots, x_k \) have been chosen such that \( a(x_i, x_j) > 0 > 3a(x_0) \) for all \( i \in \{1, \ldots, k\} \). Since \( 0 > a(x_k) = \sum_{x \in X} a(x, x_k) - a(x_k, y) \), there exists \( x_{k+1} \) with the property that \( a(x_k, x_{k+1}) > 0 \).

If \( x_{k+1} = x_i \) for some \( i \in \{0, \ldots, k\} \), then \( x_i, \ldots, x_{k+1} \) satisfies (a), (b), (c)(1).

If \( x_{k+1} \notin \{x_0, \ldots, x_k\} \) and \( 3a(x_{k+1}) > 0 \), then set \( N = k + 1 \).

If \( x_{k+1} \notin \{x_0, \ldots, x_k\} \) and \( 3a(x_{k+1}) < 0 \), then continue.

Since \( \{(x, y): a(x, y) > 0\} \) is a finite set, we either obtain the desired set or distinct points \( x_0, \ldots, x_N \) such that \( a(x_N) > 0 \) and \( a(x_{i-1}, x_i) > 0 \) for all \( i \in \{1, \ldots, N\} \). In the latter case a similar argument using decreasing indices can be used to complete the proof.

2.3. Given a 0-chain \( \beta \) such that \( \sum_{x \in X} \beta(x) = 0 \), we denote by \( M_\beta \) the number

\[
\inf \{ M(\alpha): \alpha \text{ is a } 1\text{-pchain with } \partial \alpha = \beta \}.
\]

Observe that \( M_\beta \) is finite since one can construct a 1-pchain with boundary \( \beta \) by adding an appropriate number of characteristic functions \([x, y]\) where \( \beta(x) < 0 < \beta(y) \). Note also that \( M_\beta = \inf \{ M(\alpha_0 \oplus \xi): \xi \text{ is a cycle} \} \), where \( \alpha_0 \) is any 1-pchain with \( \partial \alpha_0 = \beta \).

2.4. THEOREM. If \( \beta \) is a 0-chain in \( X \) such that \( \sum \beta(x) = 0 \), then there exists a 1-pchain \( \alpha_0 \) with \( \partial \alpha_0 = \beta \) and \( M(\alpha_0) = M_\beta \).

PROOF. For any 1-pchain \( \alpha \) with \( \partial \alpha = \beta \) one can find \( \sigma_i, i = 1, \ldots, l \), as in 2.2. Now define

\[
\gamma_i = \begin{cases} [x, y] & \text{if } \partial \sigma_i = \delta_x - \delta_y, \\ 0 & \text{if } \partial \sigma_i = 0, \end{cases}
\]

and \( \gamma = \bigoplus_{i=1}^l \gamma_i = \sum_{i=1}^l \gamma_i \). Then \( \partial \gamma = \sum_{i=1}^l \partial \sigma_i = \alpha \) and by the triangle inequality

\[
M(\gamma) = \sum_{i=1}^l M(\gamma_i) \leq \sum_{i=1}^l M(\sigma_i) = M(\alpha).
\]

Observing that \( \gamma(x, y) > 0 \) implies \( \beta(x) < 0 < \beta(y) \) we conclude

\[
M_\beta = \inf \{ M(\gamma): \partial \gamma = \beta \text{ and } \gamma(x, y) > 0 \text{ implies } \beta(x) < 0 < \beta(y) \}.
\]

The infimum is attained, since the latter set is finite.

2.5. THEOREM. For any positive integer \( k \) and 0-chain \( \beta \) in \( X \) with \( \sum_{x \in X} \beta(x) = 0 \), the equality \( M_{k\beta} = kM_\beta \) holds.

PROOF. If \( \alpha \) is a 1-pchain, then \( \partial k\alpha = k\partial \alpha \) and \( M(k\alpha) = kM(\alpha) \) from which \( M_{k\beta} \leq kM_\beta \) follows.

For the reverse inequality, we suppose \( \alpha \) is a 1-pchain such that \( \partial \alpha = \beta \) and \( M(\alpha) = M_\beta \). Given any cycle \( \xi \), we use 2.2 to decompose \( \xi \) into a sum of simple cycles \( \sigma_i, i = 1, \ldots, l \) and \( (x, y) \in X \times X \), let

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\[ s_i(x, y) = \begin{cases} -1 & \text{if } (\alpha(x, y) - \alpha(y, x))(\sigma_i(x, y) - \sigma_i(y, x)) < 0, \\ 1 & \text{otherwise.} \end{cases} \]

Since
\[
\alpha \oplus \sigma_i(x, y) + \alpha \oplus \sigma_i(y, x) = |\alpha(x, y) - \alpha(y, x) + \sigma_i(x, y) - \sigma_i(y, x)|
\]
\[= |\alpha(x, y) - \alpha(y, x)| + s_i(x, y)|\sigma_i(x, y) - \sigma_i(y, x)|,
\]
we have
\[
M(\alpha \oplus \sigma) = M(\alpha) + \frac{1}{2} \sum_{(x, y) \in X \times X} s_i(x, y)|\sigma_i(x, y) - \sigma_i(y, x)||d(x, y)|.
\]
By the mass minimality of \(\alpha\), we conclude
\[
\sum_{(x, y) \in X \times X} s_i(x, y)|\sigma_i(x, y) - \sigma_i(y, x)|d(x, y) > 0.
\]
For each \((x, y) \in X \times X\) let \(I(x, y) = \{i: s_i(x, y) < 0\}\). Then
\[
ka \oplus \sum_{i=1}^I \sigma_i(x, y) + ka \oplus \sum_{i=1}^I \sigma_i(y, x)
\]
\[= |ka(x, y) - ka(y, x) + \sum_{i=1}^I \sigma_i(x, y) - \sigma_i(y, x)|
\]
\[= |ka(x, y) - ka(y, x) + \sum_{i \in I_{(x, y)}} \sigma_i(x, y) - \sigma_i(y, x) + \sum_{i \notin I_{(x, y)}} \sigma_i(x, y) - \sigma_i(y, x)|
\]
\[= |ka(x, y) - ka(y, x)| + \sum_{i \in I_{(x, y)}} |\sigma_i(x, y) - \sigma_i(y, x)|
\]
\[> |ka(x, y) - ka(y, x)| + \sum_{i=1}^I s_i(x, y)|\sigma_i(x, y) - \sigma_i(y, x)|.
\]
Hence
\[
M(ka \oplus \xi) > M(ka) + \frac{1}{2} \sum_{(x, y) \in X \times X} \sum_{i=1}^I s_i(x, y)|\sigma_i(x, y) - \sigma_i(y, x)||d(x, y)|
\]
\[> M(ka) = kM_\beta.
\]
Thus the inequality \(M_\beta > kM_\beta\) is established.

3. Invariant solutions to Plateau problems.

3.1. Let \(G\) be a finite group acting on \(X\) so that \(d(g(x), g(y)) = d(x, y)\) for all \(x, y \in X\) and \(g \in G\). Let \(G(x)\) denote the orbit of \(x\) and \(G_x\) denote the isotropy
subgroup of $G$ at $x$. Identifying orbits of $G$ to points gives the orbit space $Z$. Let $\pi: X \to Z$ be the canonical projection. For each $u, v \in Z$ define

$$d'(u, v) = \min \{d(x, y) : x \in \pi^{-1}(u), y \in \pi^{-1}(v)\}.$$ 

The 0-chain $\beta$ and the 1-chain $\alpha$ are called invariant if $\beta(gx) = \beta(x)$ and $\alpha(g(x), g(y)) = \alpha(x, y)$ for all $x, y \in X$.

3.2. Theorem. If $\beta$ is a $G$-invariant 0-chain with $\sum_{x \in X} \beta(x) = 0$ and the order of $G$ divides $\beta(x)$ card $G(x)$ for each $x \in X$, then there exists a $G$-invariant 1-pchain $\alpha$ such that $\partial \alpha = \beta$ and $M(\alpha) = M_\beta$.

Proof. Let $r$ be the order of $G$. Define $\beta' : Z \to Z$ by

$$\beta'(u) = \frac{1}{r} \sum_{x \in \pi^{-1}(u)} \beta(x).$$

Since $\beta$ is $G$-invariant and $r$ divides $\beta(x)$ card $G(x)$, $\beta'$ is a 0-chain in $Z$. If $\gamma$ is a 1-pchain in $X$ with $\partial \gamma = \beta$, then one can define the 1-pchain in $Z$ by

$$\gamma'(u, v) = \sum_{(x, y) \in \pi^{-1}(u) \times \pi^{-1}(v)} \gamma(x, y).$$

Then $\partial \gamma' = r \beta'$ and $M(\gamma') < M(\gamma)$. Hence by 2.5, $M_\beta > M_r \beta' = rM_\beta$.

Now choose a 1-pchain $\alpha'$ such that $\partial \alpha' = \beta'$ and $M(\alpha') = M_\beta$. For each $(u, v) \in Z \times Z$ such that $\alpha'(u, v) > 0$, we find $x_u \in \pi^{-1}(u)$ and $y_{u,v} \in \pi^{-1}(v)$ such that $d(x_u, y_{u,v}) = d'(u, v)$. Let $\alpha$ be the 1-pchain in $X$ given by

$$\alpha = \sum_{(u, v) \in Z \times Z} \sum_{g \in G} \alpha'(u, v)[g(x_u), g(y_{u,v})].$$

Then

$$M(\alpha) = \sum_{(u, v) \in Z \times Z} \sum_{g \in G} \alpha(g(x_u), g(y_{u,v})) \cdot d(x_u, y_{u,v})$$

$$= \sum_{(u, v) \in Z \times Z} \sum_{g \in G} \alpha'(u, v) \cdot d'(u, v) = rM(\alpha'),$$

and

$$\partial \alpha = \sum_{(u, v) \in Z \times Z} \sum_{g \in G} \alpha'(u, v) \delta_{gw_u} - \sum_{(u, v) \in Z \times Z} \sum_{g \in G} \alpha'(u, v) \delta_{gw_u}$$

$$= \sum_{u \in Z} \left( \sum_{v \in Z} \alpha'(v, u) - \alpha'(u, v) \right) \sum_{g \in G} \delta_{gw_u}$$

$$= \sum_{u \in Z} \beta'(u) \sum_{g \in G} \delta_{gw_u},$$

where $w_u$ is any point in $\pi^{-1}(u)$. Since $(\text{card } G(x))(\text{card } G_x) = r$, we have that

$$\partial \alpha(x) = \beta'(\pi(x)) \cdot \text{card } G_x = \beta(x).$$

Hence $M_\beta < M(\alpha) = rM(\alpha') = rM_\beta < M_\beta$.

3.3. For an explanation of the terms and notation used in the following corollary, we refer the reader to [F1, pp. 670, 671].
Corollary. Let $X$ be a complete Riemannian manifold and $G$ be a finite group of isometries of $X$. If $B$ is a $G$-invariant $0$-dimensional rectifiable current which is a boundary and $r$ divides $\Theta(\|B\|, x)$ card $G(x)$ for all $x \in X$, then there exists a $G$-invariant $T \in I^1(X)$ such that $\partial T = B$ and

$$M(T) = \inf\{M(R) : R \in I^1(X) \text{ and } \partial R = B\}.$$ 

Proof. Repeat the argument of 3.2 except in the definition of $\alpha$ use $g \# L(x_u, y_{u, v})$ where $L(x_u, y_{u, v})$ is any oriented length minimizing arc from $x_u$ to $y_{u, v}$.

References


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