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FIXED POINTS IN NONCONVEX DOMAINS

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ABSTRACT. A lemma of Janos is used to prove that nonexpansive self-maps which "shrink" a compact set X away from its boundary in $\overline{\text{co}}\, X$ have fixed points in X. The lemma is further employed to derive a version, for nonexpansive maps on star-shaped sets, of Janos and Solomon's fixed-point theorem for continuous maps on spaces having an attractor for compact sets.

Janos [2], using a result of A. D. Wallace [4, Theorem 1] in the theory of topological semigroups arrived at the following lemma.

LEMMA 1 (JANOS [2]). *Let (X, d) be a compact metric space and $f\colon X \to X$ with $\{f^n | n \geq 1\}$ equicontinuous. Then there exists a retraction $r\colon X \to C_f$ where C_f denotes the core of f; i.e. $C_f = \bigcap \{f^n(X) | n \geq 1\}$.*

A particularly significant feature in the proof of this lemma is the demonstration that the retraction r of X onto C_f is in the closure, in the pointwise topology, of the set $\Gamma_n = \{f^m | m \geq n\}$ for each $n \geq 1$ (cf. [2] and [3]). This enables us to state a version of the lemma for nonexpansive maps.

LEMMA 2. *Let T be a nonexpansive self-map of a compact metric space (X, d). Then there exists a nonexpansive retraction $r\colon X \to C_T$ where*

$$C_T = \bigcap \{T^n(X) | n \geq 1\}.$$

PROOF. The family $\{T^n\}$ is equicontinuous, for given any $\varepsilon > 0$ if we merely choose $\delta = \varepsilon$ then $d(x, y) < \delta$ implies that $d(T^n x, T^n y) \leq d(x, y) < \varepsilon$ for all $n \geq 1$. Now the retraction r guaranteed by Lemma 1 is the pointwise limit of mappings of the form T^n. Thus, for any $x, y \in X$ and any $\varepsilon > 0$ there exists an n such that $d(rx, T^n x) < \varepsilon$ and $d(T^n y, ry) < \varepsilon$. Hence

$$\begin{aligned} d(rx, ry) &\leq d(rx, T^n x) + d(T^n x, T^n y) + d(T^n y, ry) \\ &< d(T^n x, T^n y) + 2\varepsilon \leq d(x, y) + 2\varepsilon. \end{aligned}$$

Since ε was arbitrary this shows $d(rx, ry) \leq d(x, y)$ and thus r is a nonexpansive mapping.

A normed linear space X is said to be strictly convex if whenever $x, y \in X$ with $x \neq y$, $\|x\| = \|y\| = S$ and $0 < \alpha < 1$, then $\|\alpha x + (1 - \alpha)y\| < S$. A useful property of strictly convex spaces is that for any three noncollinear points $x, y, z \in X$, the triangle inequality is strict; i.e. $d(x, z) < d(x, y) + d(y, z)$.

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In the results which follow, we shall denote by $\partial' X$ the boundary of X in the closed convex hull of X.

THEOREM 1. *Let X be a compact subset of a strictly convex normed linear space E and let $T: X \to X$ be nonexpansive. If there exists an $n \geqslant 1$ such that $T^n X \cap \partial' X = \varnothing$ then T has a fixed point in X.*

PROOF. As before let $C_T = \bigcap_{n=1}^{\infty} T^n X$. By Lemma 2 there exists a nonexpansive retraction $r: X \to C_T$. Suppose C_T is not convex. Then there exists $x, y \in C_T$ such that $\overline{xy} = \{\alpha x + (1-\alpha)y \mid 0 \leqslant \alpha \leqslant 1\}$ is not contained entirely in C_T. Let $a = \sup\{\beta \mid \alpha x + (1-\alpha)y \in C_T,\ 0 \leqslant \alpha \leqslant \beta\}$. Since C_T is compact, $z = ax + (1-a)y$ is in C_T. Now if $\overline{xy}$ contains no points of $X \setminus C_T$ then z is a ∂'-boundary point of X in $C_T \subseteq T^n X$ contradicting the hypothesis. Hence there exists a point $w \in \overline{xy} \cap \{X \setminus C_T\}$. But then we shall have

$$\|rx - rw\| + \|rw - ry\| \leqslant \|x - w\| + \|w - y\| = \|x - y\|$$

and, since $rx = x$, $ry = y$,

$$\|x - rw\| + \|rw - y\| \leqslant \|x - y\|.$$

But $rw \in C_T$ and so $w \neq rw$. The mapping r is nonexpansive and so rw is not collinear with x, y for if $w = bx + (1-b)y$ and $rw = cx + (1-c)y$ we shall have $b \neq c$, say, $c < b < 1$. But then

$$\|x - w\| = (1-b)\|x - y\|$$

and

$$\|rx - rw\| = \|x - rw\| = (1-c)\|x - y\|$$

and since $(1-c) > (1-b)$ we would have $\|x - w\| < \|rx - rw\|$, a contradiction. If, on the other hand, $0 < b < c$ then $\|y - w\| = b\|x - y\|$ and $\|ry - rw\| = \|y - rw\| = c\|x - y\|$ and so $\|y - w\| < \|ry - rw\|$, another contradiction. Thus, x, y, rw are not collinear. But then $\|x - rw\| + \|rw - y\| > \|x - y\|$ contradicting an earlier inequality. Thus C_T must be convex. Now $T: C_T \to C_T$ and so T has a fixed point in C_T by Schauder's Theorem.

COROLLARY 1. *Let X be a compact subset of a strictly convex normed linear space E and suppose T, g are self-maps of X such that g is continuous, T is nonexpansive and $T^n X \cap \partial' X = \varnothing$ for some $n \geqslant 1$. If $Tg = gT$ then T, g have a common fixed point in X.*

COROLLARY 2. *Let X be a compact subset of a strictly convex normed linear space E. Suppose that $\{T_\alpha\}$ is a family of commuting, nonexpansive self-maps of X and that for one $T \in \{T_\alpha\}$, there exists an $n \geqslant 1$ such that $T^n X \cap \partial' X = \varnothing$. Then the family $\{T_\alpha\}$ has a common fixed point in X.*

The proofs of these corollaries utilize standard techniques.

The suspicion that "compact" may not be replaced by "closed and bounded" in the hypotheses of Theorem 1 is easily confirmed by the following example in the Hilbert space l_2: Let e_n be the nth unit vector in l_2 and let

$$X = \bigcup_{n=1}^{\infty} B(e_n, n/(2n+1)).$$

Then X is closed and bounded. Define the mapping T: $X \to X$ by $T(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots)$. Then T is nonexpansive and $TX \cap \partial' X = \varnothing$. But T has no fixed point in X.

DEFINITION. Let X be a topological space and f: $X \to X$ a map. A nonempty subset $M \subseteq X$ is called an "attractor for compact sets under f" if (a) M is compact and $fM \subseteq M$, (b) for any compact set $G \subseteq X$ and open set 0 containing M, there exists an $N = N(G, 0)$ such that $f^n G \subseteq 0$ for all $n \geqslant N$.

Janos and Solomon [3], using Lemma 1, have proven

THEOREM 2. *Let f: $G \to G$, G a closed, convex subset of a Banach space, f a mapping satisfying*:

(i) *there exists $M \subseteq G$ an attractor for compact sets under f, and*

(ii) *the family $\{f^n\}_{n=1}^{\infty}$ is equicontinuous.*

Then f has a fixed point in X.

We shall prove a result analogous to this theorem for nonexpansive maps on star-shaped sets. The proof parallels that of Janos and Solomon.

THEOREM 3. *Let S be a closed, star-shaped subset of a Banach space E and T: $S \to S$ nonexpansive. If there exists $M \subseteq S$, an attractor for compact sets, then T has a fixed point in S.*

PROOF. Let p be a star-point of S and define the closed star-hull of M to be $\bar{\text{sh}}\, M = \text{cl}\{tp + (1-t)x | 0 \leqslant t \leqslant 1, x \in M\}$. M is compact, so $M \cup \{p\}$ is compact. Now the convex hull of a compact set is compact, so $\overline{\text{co}}[M \cup \{p\}]$ is compact. But $\bar{\text{sh}}\, M \subseteq \overline{\text{co}}[M \cup \{p\}]$ so $\bar{\text{sh}}\, M$ is compact. Let

$$X = \text{cl}\left\{\bigcup T^n(\bar{\text{sh}}\, M) | n \geqslant 0\right\}.$$

Since $T[T^n(\bar{\text{sh}}\, M)] = T^{n+1}(\bar{\text{sh}}\, M)$ for all $n \geqslant 0$ and T is continuous, T: $X \to X$. X is compact, for if $0_{\varepsilon/2}$ is the $\varepsilon/2$-parallel set of M (i.e. $0_{\varepsilon/2} = \bigcup_{x \in M} S(x, \varepsilon/2)$) there exists an N such that $n \geqslant N$ implies $T^n(\bar{\text{sh}}\, M) \in 0_{\varepsilon/2}$. But M compact implies that M has a finite $\varepsilon/2$-mesh. Thus $0_{\varepsilon/2}$ has a finite ε-mesh. So $\bigcup_{n \geqslant N} T^n(\bar{\text{sh}}\, M)$ has a finite ε-mesh and since $\bigcup_{n<N} T^n(\bar{\text{sh}}\, M)$ is compact, X is itself compact.

Now let $C_T = \bigcap_{n=0}^{\infty} T^n(X)$. C_T is nonempty and compact.

$$TC_T = T \bigcap_{n=0}^{\infty} T^n(X) = \bigcap_{n=0}^{\infty} T^{n+1}(X) = C_T,$$

so by the definition of M, $C_T \subseteq M$. By Lemma 2 there exists a nonexpansive retraction r: $X \to C_T$. The map Tr: $\bar{\text{sh}}\, M \to C_T$ is a nonexpansive self-map of a compact star-shaped set $\bar{\text{sh}}\, M$ and so by Dotson's theorem [1] has a fixed point $x \in \bar{\text{sh}}\, M$. Now $r(x) \in C_T$ and T: $C_T \to C_T$ so $x = Tr(x) \in C_T$. But then $r(x) = x$ and thus $Tx = x$.

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