ON OPERATOR RANGES

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Abstract. If \( f \) is any vector-valued bounded function defined on open set \( D \) of the complex plane, and \( T \) is any bounded linear operator on a Hilbert space such that \( \sigma_R(T^*) \) is empty and if \( (T - zI)f(z) = x \) for all \( z \) in \( D \) then \( f \) is analytic.

Let \( T \) be a bounded linear operator on a Hilbert space \( H \). Let \( f \) be an \( H \)-valued function defined on an open set \( D \) of the complex plane. Suppose \( (T - zI)f(z) = x \) for all \( z \) in \( D \) and for a fixed \( x \) in \( H \). The question arises: What type of conditions on the operator \( T \) and on the function \( f \) will be sufficient to insure that \( f \) is analytic on \( D \)? This question has been implicitly discussed in Clancey [1], Johnson [4], Putnam [5], [6], Radjabalipour [7], and Stampfli-Wadhwa [9], [10], under various conditions related to normality on \( T \). One of the main results is: If \( T \) is a hyponormal operator \( (TT^* < T^*T) \) and \( f \) is any function satisfying the above equation then \( f \) is analytic in \( D \). See [1]. On the other hand, there is a cohyponormal operator \( T \), \( (T^*) \) is hyponormal) and a bounded function \( f \) satisfying the above equation which fails to be analytic on \( D \). See [6].

In this note, by a simple argument, we shall show that if \( (T - zI)^2f(z) = x \) for all \( z \) in \( D \) and if \( f \) is bounded on \( D \) and \( \sigma_R(T^*) = \emptyset \) (range of \( T^* - zI \) is dense in \( H \) for all \( z \)) then \( f \) is analytic. We shall use this result to give an alternative proof of a classical result of Stampfli [8] about quadratically hyponormal operators.

The following lemma is implicitly contained in most of the references mentioned previously. We include it for the sake of completeness.

**Lemma.** Let \( (T - zI)f(z) = x \) for all \( z \) in \( D \) be such that \( f \) is bounded and \( \sigma_R(T^*) = \emptyset \) then \( f \) is weakly continuous.

**Proof.** For any \( z \) and \( z_0 \) in \( D \),

\[
(f(z) - f(z_0), (T^* - z_0I)y) = ((T - z_0I)(f(z) - f(z_0)), y)
= (z - z_0)(f(z), y)
\]

for all \( y \) in \( H \).

Since \( f \) is bounded and range of \( (T^* - z_0I) \) is dense in \( H \), it follows that \( f \) is weakly continuous.

**Theorem.** Let \( g \) be a bounded vector-valued function such that \( (T - zI)^2g(z) = x \) for all \( z \) in \( D \) and let \( \sigma_R(T^*) = \emptyset \). Then \( g \) is analytic on \( D \).

**Proof.** Let \( f(z) = (T - zI)g(z) \). Then \( (T - zI)f(x) = x \) for all \( z \) in \( D \), \( f \) is bounded and hence weakly continuous on each bounded subset \( D_0 \) of \( D \). Now for
any \( z \) and \( z_0 \) in \( D_0^* \),
\[
\left( \frac{(f(z) - f(z_0))}{(z - z_0)}, (T^* - z_0 I)y\right) = (f(z), y),
\]
and
\[
\lim_{z \to z_0} \left( \frac{(f(z) - f(z_0))}{(z - z_0)}, (T^* - z_0 I)y\right) = (f(z_0), y)
\]
\[
= \left( (T - z_0 I)g(z_0), y\right) = (g(z_0), (T^* - z_0 I)y)
\]
for all \( y \) in \( H \).

Since range of \( (T^* - z_0 I) \) is dense in \( H \), \( f \) is analytic and \( f'(z) = g(z) \) for all \( z \) in \( D \).

Let \( T \) be a quadratically hyponormal operator \((aT^2 + bT + cI \text{ is hyponormal for all complex numbers } a, b \text{ and } c)\). Let \( \rho(T, x) \) be the local resolvent of the vector \( x \) with respect to the operator \( T \) (see Dunford and Schwartz [3, p. 1935]).

**Corollary (Stampfli [8]).** If \( T \) is a quadratically hyponormal operator with \( \sigma_p(T) = \emptyset \) then \( \rho(T; x) \subset \rho(T^*, x) \). (The bar denotes the complex conjugate of the set.)

**Proof.** Let \( z_0 \in \rho(T, x) \); thus there exists an analytic function defined on a bounded set \( D \) containing \( z_0 \) such that \( (T - zI)f(z) = x \). Since \( f \) is analytic, a simple computation shows that \( (T - zI)^2f^2(z) = x \) for all \( z \) in \( D \). Since \( T \) is quadratically hyponormal, \( (T - zI)^2(T^* - zI)^2 < (T^* - zI)^2(T - zI)^2 \). By Douglas [2], there is a contraction \( K(z) \) such that \( (T - zI)^2 = (T^* - zI)^2K(z) \). Consequently, \( (T^* - zI)^2g(z) = x \) where \( g(z) = K(z)f(z) \). Thus \( g(z) \) is bounded for \( z \in D \). Using the Theorem we conclude that \( g(z) \) is analytic for all \( z \) in \( \overline{D} \) and \( (T^* - zI)(T^* - zI)g(z) = x \) for all \( z \) in \( \overline{D} \); thus \( z_0 \in \rho(T^*, x) \).

**References**


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