PREDICTION $n$ UNITS OF TIME AHEAD$^1$

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Abstract. The purpose of this note is to give a simple expression in terms of $w$ of the quantities

$$
\rho_n(w) = \inf_{f} \int_{0}^{2\pi} |1 + e^{i \omega f}|^2 w \, d\theta / 2\pi \quad (n = 0, 1, 2, \ldots),
$$

where $f$ ranges over the analytic trigonometric polynomials with mean value zero and $w$ is nonnegative and summable on the circle.

Helson [2, pp. 21–22] said that it is unreasonable to expect to have a simple expression in terms of $w$ for the quantities $\rho_n$ except $n = 0$. $\rho_0(w) = \exp \int_0^{2\pi} \log w \, d\theta / 2\pi$ is the famous Szegö theorem. We may assume $\log w$ is summable, because otherwise $\rho_n(w) = 0$ for all $n$.

Theorem. Let $w$ be nonnegative and summable on the circle. Suppose $\log w$ is summable and

$$
\log w(\theta) \sim \sum_{j = -\infty}^{\infty} a_j e^{i j \theta}.
$$

Then

$$
\rho_n(w) = \inf_{f} \int_{0}^{2\pi} |1 + e^{i \omega f}|^2 w \, d\theta / 2\pi
$$

$$
= e^{a_0} \times \sum_{j = 0}^{n} \prod_{m = 1}^{n} \left| \frac{a_m}{m!} \right|^2
$$

where $n > 1$ and $f$ ranges over the analytic trigonometric polynomials with mean value zero and $\Sigma'$ is the summation of all permutations of nonnegative integers $m_1, m_2, \ldots, m_n$ with $m_1 + 2m_2 + \cdots + nm_n = j$ for each $j$.

Proof. Set $g_1 = \sum_{i = 1}^{n} a_i z^i$ and $g_2 = a_0 / 2 + \sum_{i = n+1}^{\infty} a_n z^i$, then their radial limits satisfy $w(\theta) = |\exp g_1(e^{i \theta})|^2 |\exp g_2(e^{i \theta})|^2$ a.e. $\theta$. $\exp(g_1 + g_2)$ is an outer function [3, p. 61] and so $\exp g_2$ is outer. Hence, if we note that there exist positive numbers $e$ and $M$ with $0 < e < |\exp g_1(e^{i \theta})|^2 < M < \infty$, as in the proof of Szegö's theorem.
\[
\rho_n(w) = \inf \int |1 + e^{ik\theta}|^2 |\exp g_2|^2 |\exp g_1|^2 \, d\theta / 2\pi
\]

\[
= \inf \int |\exp g_2 + e^{ik\theta} \exp g_2|^2 |\exp g_1|^2 \, d\theta / 2\pi
\]

\[
= \inf \int |e^{a_w/2} + e^{ik\theta}|^2 |\exp g_1|^2 \, d\theta / 2\pi
\]

\[
= e^{a_0} \inf \int |1 + e^{ik\theta}|^2 |\exp g_1|^2 \, d\theta / 2\pi.
\]

Since \( \exp g_1 \) is an outer function, if the Fourier coefficients of \( \exp g_1 \) are \( \{b_j\} \), then (cf. [1, pp. 184–187], [2, p. 22])

\[
\inf \int |1 + e^{ik\theta}|^2 |\exp g_1|^2 \, d\theta / 2\pi
\]

\[
= \sum_{j=0}^{n} |b_j|^2 \exp g_1(z) = \prod_{l=1}^{n} \exp(a_l z^l) = \sum \frac{(a_1 z)^{m_1} \cdots (a_n z^n)^{m_n}}{m_1! \cdots m_n!}
\]

where the \( m_j \) range independently over nonnegative integers. This implies the theorem.

The theorem shows the following:

\[
\rho_1(w) = \inf \int_0^{2\pi} |1 + e^{ik\theta}|^2 w \, d\theta / 2\pi
\]

\[
= \exp \int_0^{2\pi} \log w \, d\theta / 2\pi \left( 1 + \int_0^{2\pi} (\log w) e^{ik\theta} \, d\theta / 2\pi \right)^2
\]

If \( w \) is nonnegative and summable, and \( \log w \) is summable, it is known (cf. [2, p. 20]) that \( w = |g|^2 \) for some outer \( g \). The theorem gives a simple expression in terms of \( w \) of the Fourier coefficients of outer function \( g \) (cf. [4]).

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REFERENCES


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