ON NOWHERE DENSE CCC P-SETS

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Abstract. We prove that no compact Hausdorff space can be covered by nowhere dense ccc P-sets. As an application it follows that if \( X \) is a compact Hausdorff space with a nonisolated \( P \)-point then \( X \times K \) is not homogeneous for any compact ccc space \( K \).

1. Introduction. All spaces under discussion are Tychonoff.

A subset \( B \) of a space \( X \) is called a P-set whenever the intersection of countably many neighborhoods of \( B \) is again a neighborhood of \( B \). It is known that no compact space of \( \pi \)-weight \( \omega_1 \) can be covered by nowhere dense P-sets [KvMM]. In addition, there is a compact space of weight \( \omega_2 \) which can be covered by nowhere dense P-sets [KvMM]. In this note we will show that no compact space can be covered by nowhere dense ccc P-sets. As a consequence it follows that if \( X \) is a compact space with a nonisolated \( P \)-point then \( X \times K \) is not homogeneous for any compact ccc space \( K \).

2. Independent matrices. Let \( X \) be a space. An indexed family \( \{A^i_j : i \in I, j \in J\} \) is called an \( I \) by \( J \) independent matrix for \( X \) provided that

(a) each \( A^i_j \) is an open \( F_{n \alpha} \);
(b) if \( i \in I \) and \( j_0, j_1 \) are distinct elements of \( J \) then \( A_{i_0}^i \cap A_{i_1}^i = \emptyset \);
(c) if \( F \subset I \) is finite and \( \varphi : F \rightarrow J \) then \( \bigcap_{i \in F} A_{\varphi(i)}^i \neq \emptyset \).

This concept, in a slightly different form, is due to Kunen.

In [vM4] it was shown that each compact space in which each nonempty \( G_\delta \) has nonempty interior contains an \( \omega_1 \) by \( \omega_1 \) independent matrix. We need a generalization of this result. As usual, a space is called ccc if each pairwise disjoint collection of nonempty open sets is countable. A space is nowhere ccc if no point has a ccc neighborhood.

2.1. Theorem. Suppose that \( X \) is nowhere ccc. Then \( X \) contains an \( \omega_1 \) by \( \omega_1 \) independent matrix.

Proof. For each finite subset \( F \subset \omega_1 \) (possibly empty) we will define an open \( F_{n \alpha} \), \( C_F \subset X \), such that

(i) \( C_{F \cup (\alpha)} \subset C_F \) for all \( \max F < \alpha < \omega_1 \);
(ii) \( C_{F \cup (\alpha)} \cap C_{F \cup (\beta)} = \emptyset \) if \( \max F < \alpha < \beta < \omega_1 \)

(as usual, an ordinal is the set of smaller ordinals; we define \( \max \emptyset = -1 \)).

We will induct on the cardinality of \( F \). Define \( C_{\emptyset} = X \).
Suppose that we have defined $C_F$ for all $F \subseteq \omega_1$ of cardinality $n$. Let $\{C_{F \cup \{a\}}: \max F < a < \omega_1\}$ be a “faithfully indexed” collection of pairwise disjoint non-empty open $F_s$'s of $C_F$. This completes the induction.

**Fact.** $C_F \cap C_G = \emptyset \rightarrow (F \subseteq G) \lor (G \subseteq F)$.

We induct on the cardinality of $|F| + |G|$. If $|F| + |G| = 1$ then there is nothing to prove. Suppose that we have proved the Fact for all finite sets $F, G \subseteq \omega_1$ satisfying $|F| + |G| < n - 1$. Now take finite sets $S, T \subseteq \omega_1$ so that $|S| + |T| < n$. Define $S' = S - \{\max S\}$. By (i) we have that $C_S \subseteq C_{S'}$; and consequently $C_{S'} \cap C_T = \emptyset$. By induction hypothesis, $S' \subseteq T$ or $T \subseteq S'$. If $T \subseteq S'$ then we are done, so we may assume that $S' \subseteq T$. Define $T' = T - \{\max T\}$. By precisely the same argumentation we may conclude that $T' \subseteq S$. Then clearly

$$(S \cap T) \cup \{\max S\} = S \quad \text{and} \quad (S \cap T) \cup \{\max T\} = T.$$  

If $\max S \in T$ or $\max T \in S$ then there is nothing to prove. So assume that this is not true. Then by (ii) we have that $C_S \cap C_T = \emptyset$, which is a contradiction.

Let $f: \omega_1 \to \omega_1 \times \omega_1$ be onto and one-to-one. Define $U^a_\beta = \{C_{F \cup \{f^{-1}(\langle a, \beta \rangle)\}}: \max F < f^{-1}(\langle a, \beta \rangle)\}$ and $f[F] \cap \{(\alpha) \times \omega_1\} = \emptyset$. Notice that $C_{f^{-1}(\langle a, \beta \rangle)} \subseteq U^a_\beta$. We claim that $\{U^a_\beta: a, \beta < \omega_1\}$ is an $\omega_1$ by $\omega_1$ independent matrix for $X$. First observe that each $U^a_\beta$ is an open $F_s$ being the union of at most countably many open $F_s$'s.

Now, let us assume that $U^a_\beta \cap U^a_\gamma = \emptyset$ for some $\beta \neq \gamma$. Without loss of generality assume that $f^{-1}(\langle a, \beta \rangle) < f^{-1}(\langle a, \gamma \rangle)$. There are finite sets $F_0, F_1 \subseteq \omega_1$ so that

(a) $C_{F_0 \cup \{f^{-1}(\langle a, \beta \rangle)\}} \cap C_{f^{-1}(\langle a, \gamma \rangle)} = \emptyset$;
(b) $\max F_0 < f^{-1}(\langle a, \beta \rangle)$ and $f[F_0] \cap \{(\alpha) \times \omega_1\} = \emptyset$;
(c) $\max F_1 < f^{-1}(\langle a, \gamma \rangle)$ and $f[F_1] \cap \{(\alpha) \times \omega_1\} = \emptyset$.

Since $f^{-1}(\langle a, \gamma \rangle) \notin F_0 \cup \{f^{-1}(\langle a, \beta \rangle)\}$, by the Fact, $F_0 \cup \{f^{-1}(\langle a, \beta \rangle)\} \notin F_1 \cup \{f^{-1}(\langle a, \gamma \rangle)\}$. Therefore $f^{-1}(\langle a, \beta \rangle) \in F_1$, since $f^{-1}(\langle a, \beta \rangle) \neq f^{-1}(\langle a, \gamma \rangle)$. However, this contradicts (c).

Take $\alpha_1, \ldots, \alpha_n < \omega_1$ so that $\alpha_i \neq \alpha_j$ for $i \neq j$. In addition, take $\beta_i < \omega_1$ ($i < n$) arbitrarily. Put $\gamma_i = f^{-1}(\langle \alpha_i, \beta_i \rangle)$ and without loss of generality assume that $\gamma_1 < \gamma_2 < \cdots < \gamma_n$. Then $C_{\langle \gamma_1, \ldots, \gamma_n \rangle} \subseteq U^a_{\beta_1} \cap \cdots \cap U^a_{\beta_n}$, and since $C_{\langle \gamma_1, \ldots, \gamma_n \rangle} \neq \emptyset$ we find that $U^a_{\beta_1} \cap \cdots \cap U^a_{\beta_n} \neq \emptyset$.  

3. The first application. A point $x \in X$ is called a weak $P$-point provided that $x \notin F$ for each countable $F \subseteq X - \{x\}$. Kunen [K] proved that there is a weak $P$-point in $\omega^* (= \beta\omega \setminus \omega)$. Subsequently van Mill [vM_1] proved that there is a weak $P$-point in each compact $F$-space of weight $2^\omega$ in which each nonempty $G_\delta$ has nonempty interior (an $F$-space is a space in which each cozero set is $C^*$-embedded). Bell [B] has since shown that the weight condition is superfluous. Using Theorem 2.1 by precisely the same technique as in [vM_1] we obtain the following generalization.

3.1. **Theorem.** Each compact nowhere ccc $F$-space contains a weak $P$-point.  

4. The main result. In this section we derive our main result. The techniques of proof used in the following lemma is the same as in [vM_1], [vM_2].
4.1. Lemma. No compact nowhere ccc space can be covered by ccc P-sets.

Proof. Let \( X \) be a compact nowhere ccc space. Clearly \( X \) is not finite, so there is a collection \( \{ V_n : n < \omega \} \) of (faithfully indexed) pairwise disjoint nonempty open \( F_n \) subsets of \( X \). For each \( n < \omega \) let \( \{ U^i_{\alpha}(n) : \alpha < \omega_1, i < \omega \} \) be an \( \omega_1 \) by \( \omega \) independent matrix for \( V_n \) (Theorem 1.1). Notice that each \( U^i_{\alpha}(n) \) is an open \( F_n \) of \( X \). Put \( \mathcal{F} = \{ A \subset X : \forall n < \omega \, \forall i < n \, \exists \alpha < \omega_1 \text{ such that } U^i_{\alpha}(n) \subset A \} \). It is clear that \( \mathcal{F} \) has the finite intersection property, so there is an \( x \in \bigcap_{F \in \mathcal{F}} F \). We claim that \( x \notin K \) for each ccc P-set \( K \). Indeed, let \( K \subset X \) be any ccc P-set. Since \( K \) is ccc for each \( n < \omega \) and for each \( i < n \) there is an \( \alpha(n, i) < \omega_1 \) so that

\[
U^i_{\alpha(n,i)}(n) \cap K = \emptyset.
\]

Put \( F = \bigcup_{n<\omega} \bigcup_{i<n} U^i_{\alpha(n,i)}(n) \). Then \( F \in \mathcal{F} \) and \( F \) is an open \( F_n \) being the union of countably many open \( F_n \)'s. Also, \( F \cap K = \emptyset \). Since \( K \) is a P-set, it also follows that \( F \cap K = \emptyset \). We conclude that \( x \notin K \). 

We now come to our main result.

4.2. Theorem. No compact space can be covered by ccc nowhere dense P-sets.

Proof. Let \( X \) be a compact space and suppose that \( X \) can be covered by ccc nowhere dense P-sets. Let \( U \subset X \) be nonempty and open and suppose that \( U \) is ccc. Let \( B \) be a nowhere dense P-set meeting \( U \). Since \( B \cap U \) is nowhere dense in \( U \) the fact that \( U \) is ccc implies that there is a countable family \( \mathcal{G} \) of compact subsets of \( U - B \) so that \( \bigcup \mathcal{G} \) is dense in \( U \). However, this is impossible since \( B \) is a P-set. So \( U \) is not ccc. But now the assumption that \( X \) can be covered by ccc nowhere dense P-sets contradicts Lemma 4.1. 

5. Another application. A space \( X \) is called homogeneous provided that for all \( x, y \in X \) there is an autohomeomorphism \( \varphi \) from \( X \) onto \( X \) mapping \( x \) onto \( y \). It is well known that although \( X \) is not homogeneous the product \( X \times K \) can be homogeneous for certain \( K \) (for example, let \( X \) be a convergent sequence and let \( K \) be the Cantor set). This makes the following straightforward corollary to Theorem 4.2 of some interest.

5.1. Corollary. Let \( X \) be a compact space having a nonisolated P-point. Then \( X \times K \) is not homogeneous for any compact ccc nonempty space \( K \).

Proof. Let \( x \) be a nonisolated P-point of \( X \). Then \( \{ x \} \times K \) is a ccc nowhere dense P-set of \( X \times K \). Take any \( \langle x, y \rangle \in \{ x \} \times K \). By Theorem 4.2 there is a point \( \langle p, q \rangle \in X \times K \) so that \( \langle p, q \rangle \notin E \) for any nowhere dense ccc P-set \( E \subset X \times K \). It is clear that no autohomeomorphism of \( X \times K \) can map \( \langle x, y \rangle \) onto \( \langle p, q \rangle \). 

6. Questions. Since there is a compact space \( X \) of weight \( \omega_2 \) which can be covered by nowhere dense P-sets (which all have to have cellularity at most \( \omega_2 \)), Theorem 4.2 suggests the following question:

6.1. Question. Is there a compact space \( X \) which can be covered by nowhere dense P-sets of cellularity at most \( \omega_1 \)?
Since Frankiewicz and Mills [FM] have shown that Con(ZFC + $\omega^*$ can be covered by nowhere dense P-sets) the question naturally arises whether it is consistent that $\omega^*$ can be covered by nowhere dense P-sets of cellularity at most $\omega_1$.

Let us answer this question.

6.2. **Proposition.** $\omega^*$ cannot be covered by nowhere dense P-sets of cellularity at most $\omega_1$.

**Proof.** Under CH the result follows from [KvMM]. So assume $\neg$CH. Kunen [K] proved that (in ZFC) there is a $2^\omega$ by $2^\omega$ independent matrix of clopen subsets of $\omega^*$. Since $\omega_1 < 2^\omega$ we can use the same proof as in Lemma 4.1 to get a point $x \in \omega^*$ so that $x \not\in B$ for any P-set $B$ of cellularity at most $\omega_1$. □

Let us finally notice that Proposition 5.1 suggests the following question.

6.3. **Question.** Let $X$ be a compact space having a nonisolated P-point and let $K$ be compact. Is $X \times K$ not homogeneous?

**References**


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