ANOTHER \(q\)-EXTENSION OF THE BETA FUNCTION

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Abstract. Another \(q\)-extension of the beta function is given. This one has a special case that is a symmetric extension of the symmetric beta distribution.

1. Introduction. The beta function
\[
B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1 - t)^{\beta-1} \, dt
\]
was reduced to a function of one variable by Euler when he proved that
\[
B(\alpha, \beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha + \beta).
\]
Changes of variable can be made in (1), so other integrals can be evaluated. One trivial change is to
\[
\int_{-c}^{d} \left(1 - \frac{x}{d}\right)^{\beta-1} \left(1 + \frac{x}{c}\right)^{\alpha-1} \, dx = \frac{(c + d)^{\alpha + \beta - 1}}{c^{\alpha - 1}d^{\beta - 1}} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}.
\]
The special case \(\alpha = \beta, c = d\) is important, for the resulting distribution function is the weight function for the ultraspherical polynomials, which contain the zonal spherical harmonics (or zonal functions) on the \(k\)-dimensional sphere.

Recently there has been a renewal of interest in basic hypergeometric, or \(q\)-series, extensions of classical results. If \(0 < q < 1\), define
\[
(a; q)_n = \frac{(aq^n; q)_n}{(q; q)_n},
\]
and
\[
\Gamma_q(x) = ((q; q)_\infty)/(q^x; q)_\infty(1 - q)^{-x}.
\]
The \(q\)-binomial theorem [1, Theorem 2.1]
\[
\binom{ax}{x}_\infty = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} x^n
\]
can be rewritten in a form which extends (1.1). Following F. H. Jackson define
\[
\int_0^d f(x) \, dq x = d(1 - q) \sum_{n=0}^{\infty} f(dq^n)q^n.
\]
Then (1.7) is essentially equivalent to

\[ \int_0^1 t^{a-1} \frac{(tq; q)_\infty}{(t; q)_\infty} \, dt = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha + \beta)}. \]  

(1.9)

One drawback with the q-integral is that changes of variable are usually not possible. However there are often extensions of the integrals that arise after a change of variable. Two extensions of the beta function put on \((0, \infty)\) as

\[ \int_0^\infty t^{a-1} \frac{\Gamma_q(\alpha + \beta)}{(1 + t)^{a+\beta}} \, dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}. \]  

(1.10)

were found by Ramanujan. This was pointed out in [3] and simple proofs were given of the following.

\[ \int_0^\infty t^{a-1} \frac{(-tq^{a+\beta}; q)_\infty}{(-t; q)_\infty} \, dt = \frac{\Gamma(\alpha)\Gamma(1 - \alpha)\Gamma_q(\beta)}{\Gamma_q(1 - \alpha)\Gamma(\alpha + \beta)}. \]  

(1.11)

and

\[ \int_0^\infty t^{a-1} \frac{(-ctq^{a+\beta}; q)_\infty}{(-ct; q)_\infty} \, dt = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)(-cq^a; q)_\infty(-q^{1-a}/c; q)_\infty}{\Gamma_q(\alpha + \beta)(-c; q)_\infty(-q/c; q)_\infty}. \]  

(1.12)

However none of these integrals contains a special case that extends the symmetric beta function in a way so that the symmetry is obvious. We were led to such an extension while studying some orthogonal polynomials [2]. This extension is of independent interest, and so will be given here.

2. An extension of the beta function.

THEOREM 1. If \(|q| < 1\) and there are no zero factors in the denominator of the integrals, then

\[ \int_{-c}^d \frac{(-qx/c; q)_\infty(qx/d; q)_\infty}{(-ax/c; q)_\infty(bx/d; q)_\infty} \, dq \, dx = \frac{(1 - q)(q; q)_\infty(ab; q)_\infty(-cd/d; q)_\infty(-d/c; q)_\infty}{(a; q)_\infty(b; q)_\infty(c + d)(-bc/d; q)_\infty(-ad/c; q)_\infty}. \]  

(2.1)

or, when \(0 < q < 1\),

\[ \int_{-c}^d \frac{(-qx/c; q)_\infty(qx/d; q)_\infty}{(-xq^a/c; q)_\infty(xq^\beta/d; q)_\infty} \, dq \, dx = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha + \beta)} \frac{cd}{c + d} \frac{(c/d; q)_\infty(-d/c; q)_\infty}{(-q^\beta c/d; q)_\infty(-q^a d/c; q)_\infty}. \]  

(2.2)
PROOF. As with the usual integral, \( I^d_c = \int_0^d f^\alpha_c \). Thus if \( I \) denotes the integral in (2.1)

\[
I = d(1 - q) \sum_{n=0}^{\infty} \frac{(-dq^{n+1}/c; q)_{\infty}(q^{n+1}; q)_{\infty}q^n}{(-adq^n/c; q)_{\infty}(bq^n; q)_{\infty}}
\]

\[+ c(1 - q) \sum_{n=0}^{\infty} \frac{(q^{n+1}; q)_{\infty}(-cq^{n+1}/d; q)_{\infty}q^n}{(aq^n; q)_{\infty}(-bcq^n/d; q)_{\infty}}
\]

\[= d(1 - q) \frac{(-dq/c; q)_{\infty}(q; q)_{\infty}}{(-ad/c; q)_{\infty}(b; q)_{\infty}} 2\Phi_1\left(\frac{-ad/c, bd}{-dq/c}; q, q\right)
\]

\[+ c(1 - q) \frac{(-cq/d; q)_{\infty}(q; q)_{\infty}}{(a; q)_{\infty}(-bc/d; q)_{\infty}} 2\Phi_1\left(\frac{-bc/d, a}{-cq/d}; q, q\right)
\]

where

\[2\Phi_1\left(\frac{a, b}{c}, q; x\right) = \sum_{n=0}^{\infty} \frac{(a; q)_n(b; q)_n}{(c; q)_n(q; q)_n} x^n.\] (2.3)

It is not possible to sum these series separately (in the general case), for if it could be done we could evaluate \( \int_0^1 (1 - t)^{\alpha - 1} (1 + t)^{\beta - 1} \, dt \). So if there is to be a hope of evaluating (2.1) the two series must be put together into a single sum. To do this recall a transformation of Heine [1, Corollary 2.3].

\[2\Phi_1\left(\frac{a, b}{c}, q; x\right) = \frac{(ax; q)_{\infty}(b; q)_{\infty}}{(x; q)_{\infty}(c; q)_{\infty}} 2\Phi_1\left(\frac{c/b, x}{ax}; q, b\right).\] (2.4)

Use of (2.4) gives

\[
I = \frac{d(1 - q)}{(1 - b)} \sum_{n=0}^{\infty} \frac{(q/a; q)_n}{(bq; q)_n} \left(-\frac{ad}{c}\right)^n
\]

\[+ c(1 - q) \sum_{n=0}^{\infty} \frac{(q/b; q)_n}{(aq; q)_n} \left(-\frac{bc}{d}\right)^n.
\]

Replace \( n \) by \( -n - 1 \) in the second sum and use

\[\frac{(a; q)_{-n}}{(b; q)_{-n}} = \frac{(bq^{-n}; q)_{-n}}{(aq^{-n}; q)_{-n}} = \frac{b^n}{a^n} \frac{(q/b; q)_n}{(q/a; q)_n}
\]

to obtain

\[
I = \frac{d(1 - q)}{(1 - b)} \sum_{n=0}^{\infty} \frac{(qa^{-1}; q)_n}{(bq; q)_n} \left(-\frac{ad}{c}\right)^n.
\]

Ramanujan summed this series as

\[
\sum_{n=0}^{\infty} \frac{(a; q)_n}{(b; q)_n} x^n = \frac{(ax; q)_{\infty}(q/ax; q)_{\infty}(q; q)_{\infty}(b/a; q)_{\infty}}{(x; q)_{\infty}(b/ax; q)_{\infty}(q/a; q)_{\infty}(q/a; q)_{\infty}}.
\] (2.5)

This is another way of writing (1.12). Using this gives

\[
I = \frac{d(1 - q)(-qd/c; q)_{\infty}(-c/d; q)_{\infty}(q; q)_{\infty}(ab; q)_{\infty}}{(-ad/c; q)_{\infty}(-bc/d; q)_{\infty}(a; q)_{\infty}(b; q)_{\infty}}.
\]

This derivation used the conditions \(|b| < |d/c| < 1/|a|\) to be able to sum (2.5). However both sides of (2.1) are meromorphic functions with at most poles of finite
order at any finite point as functions of either $a$ or $b$, and as functions of $c$ and $d$ when $c$ and $d$ are in any compact subset of the plane with the origin removed. Use of analytic continuation completes the proof of Theorem 1.

The referee asked if a $q$-extension of (1.3) exists as a Riemann integral. It does, and the most general one with a real absolutely continuous integrand known at present can be written as

$$
\frac{1}{\pi} \int_{-1}^{1} \frac{h_q(1; x)h_q(q^{1/2}; x)h_q(-1; x)h_q(-q^{1/2}; x)}{(1 - x^2)^{1/2}} dx = \frac{\Gamma_q(\alpha + \beta)\Gamma_q(\gamma + \delta)(-1; q)\infty(-q^{1/2}; q)\infty(-q; q)\infty}{\Gamma_q(\alpha + \beta + \gamma + \delta)\left[\Gamma_q(\frac{1}{2})\right]^2(-q^{a+\gamma}; q)\infty(-q^{a+\delta}; q)\infty(-q^{\beta+\gamma}; q)\infty(-q^{\beta+\delta}; q)\infty}
$$

when $0 < q < 1$, $\alpha, \beta, \gamma, \delta > 0$ and $h_q(a; x) = \prod_{n=0}^{\infty}(1 - 2axq^n + a^2q^{2n})$. This will appear in [4].

REFERENCES


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