ON INSERTION OF A CONTINUOUS FUNCTION

M. POWDERLY

Abstract. Recently E. P. Lane proved that if space X has the weak C-insertion property and satisfies another condition, then X has the strong C-insertion property. This paper establishes the converse of this result.

In a recent paper, E. P. Lane [1] established two main results about the insertion of continuous functions. The second of these is:

Theorem 3.1. Let $P_1$ and $P_2$ be C-properties and consider the following condition:

(a) If $g$ and $f$ are functions on $X$ such that $g < f$, $g$ satisfies property $P_1$, and $f$ satisfies property $P_2$, then there exists a sequence $\{A(f - g, 2^{-n})\}$ of lower cut sets for $f - g$ and there exists a sequence $\{F_n\}$ of subsets of $X$ such that

(i) $\{x/(f - g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n$, and

(ii) for each $n$ the sets $A(f - g, 2^{-n})$ and $F_n$ are completely separated.

If $X$ satisfies the weak C-insertion property for $(P_1, P_2)$ and if $X$ satisfies (a), then $X$ satisfies the strong C-insertion property for $(P_1, P_2)$. Conversely, if $X$ satisfies the strong C-insertion property for $(P_1, P_2)$ and $f - g$ satisfies the property $P_1$ (actually $P_1$ should have read $P_2$), then $X$ satisfies (a).

This paper shows that the last sentence of the above theorem can be strengthened to read:

Conversely, if $X$ satisfies the strong C-insertion property for $(P_1, P_2)$, then $X$ satisfies (a). Thus (a) is a necessary and sufficient condition for a space with the weak C-insertion property to have the strong C-insertion property.

A property $P$ defined relative to a real-valued function on a topological space is a C-property provided any constant function has property $P$ and provided the sum of a function with property $P$ and any continuous function also has property $P$. Let $P_1$ and $P_2$ be C-properties. A space $X$ is said to have the weak C-insertion property for $(P_1, P_2)$ iff for any functions $g$ and $f$ on $X$ such that $g < f$, $g$ has property $P_1$ and $f$ has property $P_2$, then there exists a continuous function $h$ on $X$ such that $g < h < f$. A space $X$ is said to have the strong C-insertion property for $(P_1, P_2)$ iff for any functions $g$ and $f$ on $X$ such that $g < f$, $g$ satisfies $P_1$ and $f$ satisfies $P_2$, then there exists a continuous function $h$ on $X$ such that $g < h < f$ and such that if $g(x) < f(x)$ for any $x$ in $X$, then $g(x) < h(x) < f(x)$. If $f$ is a real-valued function defined on a space $X$ and if

$$\{x/f(x) < r\} \subset A(f, r) \subset \{x/f(x) < r\},$$

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for a real number $r$, then $A(f, r)$ is called a lower cut set.

We now prove the result stated above:

**Proof.** Assume that $X$ satisfies the strong $C$-insertion for $(P_1, P_2)$. Let $g$ and $f$ be functions on $X$ satisfying $P_1$ and $P_2$ respectively such that $g < f$. Thus there exists $h$ continuous over $X$ such that $g < h < f$ and such that if $g(x) < f(x)$ for any $x$ in $X$, then $g(x) < h(x) < f(x)$. Now consider the functions $0$ and $f - h$. $0$ satisfies property $P_1$ and $f - h$ satisfies property $P_2$. Thus there exists function $h_1$ continuous over $X$ such that $0 < h_1 < f - h$ and if $0 < (f - h)(x)$ for any $x$ in $X$, then $0 < h_1(x) < (f - h)(x)$. We next show that

$$\{x/ (f - g)(x) > 0\} = \{x/ h_1(x) > 0\}.$$  

If $x$ is such that $(f - g)(x) > 0$, then $g(x) < f(x)$. Therefore $g(x) < h(x) < f(x)$. Thus $f(x) - h(x) > 0$ or $(f - h)(x) > 0$. Hence $h_1(x) > 0$. On the other hand, if $h_1(x) > 0$, then since $(f - h) > h_1$ and $f - g > f - h$, therefore $(f - g)(x) > 0$. For each $n$, let $A(f - g, 2^{-n}) = \{x/ (f - g)(x) < 2^{-n}\}$,

$F_n = \{x/ h_1(x) > 2^{-n+1}\},$

and

$$k_n = \sup\{\inf\{h_1, 2^{-n+1}\}, 2^{-n}\} - 2^{-n}.$$  

Since $\{x/ (f - g)(x) > 0\} = \{x/ h_1(x) > 0\}$, it follows that

$$\{x/ (f - g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n.$$  

We next show that $k_n$ is a continuous function from $X$ into $[0, 2^{-n}]$ which completely separates $F_n$ and $A(f - g, 2^{-n})$. From its definition, it is clear that $k_n$ is continuous over $X$. Let $x \in F_n$. Then, from the definition of $k_n$, $k_n(x) = 2^{-n}$. If $x \in A(f - g, 2^{-n})$, then since $h_1 < f - h < f - g$, $h_1(x) < 2^{-n}$. Thus $k_n(x) = 0$, according to the definition of $k_n$. Hence $k_n$ completely separates $F_n$ and $A(f - g, 2^{-n})$. This completes our proof.

A paper by Blatter and Seever [2] deals with closed lattice cones (of functions) on a set $X$, i.e., closed convex cones of bounded real-valued functions on $X$ which contain the constants and which are closed under the lattice operation. The necessary and sufficient conditions are presented for the “interposition” of $K$ between members of the pair $(L, M)$, $K, L, M$ being closed lattice cones. The results are given in terms of certain binary relations, called inclusions, on the power set of $X$. However, a lattice cone of functions on a set $X$ and sets with $C$-insertion properties are in general not the same.

**References**


**Department of Mathematics, William Paterson College of New Jersey, Wayne, New Jersey 07470**