A REMARKABLE SIMPLE CLOSED CURVE: REVISITED

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Abstract. It is shown that the pathology of R. H. Fox's remarkable simple closed curve is in a sense explained below more complicated than that of some examples of the well-known Fox-Artin paper.

A classical example of Fox-Artin shows that the union of two tamely embedded arcs whose intersection is a common end-point may be wildly embedded [3, Example 1.4]. Arcs formed in this way are called mildly wild. A classification theorem exists for such sets if the union is locally peripherally unknotted, LPU, also called Wilder arcs [4]. While the concepts of LPU and LU [7] have served to classify the local pathology of the examples of [3], a more delicate invariant seems necessary to relate the embedding of [5] to those in [3]. There is the possibility that an appropriately chosen arc from "The remarkable simple closed curve" might be "less wildly" embedded than the arcs in [3]. This note dispels such feasibility. To make this more precise a new embedding condition is introduced related to LPU. First, however, we review Fox's concept of almost unknotted.

A simple closed curve \( \Gamma \) in three-space \( \mathbb{R}^3 \) is called almost unknotted if there is a point \( p \) and a neighborhood \( U \) of \( p \) such that for any neighborhood \( V \) of \( p \) there is a homeomorphism \( \phi \) of \( \mathbb{R}^3 \) on \( \mathbb{R}^3 \) such that

(i) \( \phi \) is the identity on \( V \),
(ii) \( \phi|_{\Gamma \setminus U} \) is a subset of a plane.

It is clear that if \( q \) is another point and \( \Gamma \) has the same property at \( q \), then \( \Gamma \) is unknotted.

There is a local property suggested by the above. An arc \( X \) is called locally almost unknotted at \( p \) if for some neighborhood \( U \) of \( p \), no matter how small a neighborhood \( V \) of \( p \) is chosen, there is a neighborhood \( W \) of \( p \) and a homeomorphism \( \phi \) of \( \mathbb{R}^3 \) on \( \mathbb{R}^3 \) such that

(i) \( \phi = \) identity on \( W \) and
(ii) \( \phi|_{U \setminus V} \) is a subset of a plane. We abbreviate this property by LAU (locally almost unknotted).

Before introducing the next definition recall the definition of local peripheral unknottedness for a 1-manifold in \( \mathbb{R}^3 \) [7]. Let \( p \) be an interior point (boundary point) of \( X \) and \( \epsilon > 0 \). It is required that there be a topological 2-sphere \( K \) whose interior contains \( p \) such that
(i) $\text{diam} \ K < \epsilon$,
(ii) $\text{card} \ K \cap X = 2$ or 1 according as $p$ is an interior or boundary point of $X$.

See [3, Examples 1.1 or 1.2] for the failure of this property.

A 1-manifold $X \subset \mathbb{R}^3$ is called weakly peripherally unknotted at $p$ if for each $\epsilon > 0$ there is a homeomorphism $h_\epsilon$ of $\mathbb{R}^3$ on $\mathbb{R}^3$ such that

1. $h_\epsilon(p) = p$,
2. $\text{diam} \ h_\epsilon^{-1}(K) < \epsilon$,
3. $\text{card} \ A_\epsilon \cap X = 2$ or 1 according as $p$ is an interior or boundary point of $X$.

We abbreviate this property by WPU. Note “The remarkable simple closed curve” has this property at each point.

If one may take $h_\epsilon$ = identity for all $\epsilon > 0$, this becomes LPU.

As a preliminary to the main result (Theorem 2) we have the following.

**Theorem 1.** Let $X$ be an arc that is locally tame modulo $p$, $p$ an end-point of $X$. Then $X$ is LAU at $p$.

**Proof.** If the penetration index of $X$ at $p$ is 1, a sequence of space homeomorphisms can be defined carrying $X$ onto an interval. Hence $X$ is tame. In general, by the smoothing techniques of [1] or [9], $X \setminus p$ may be taken as locally polyhedral. If the other end-point of $X$ is $q$, let $X$ be ordered from $q$ to $p$. Also, let $F(p, \epsilon)$ denote the surface of a sphere of radius $\epsilon$, centered at $p$. Let $Y_\epsilon$ denote the subarc of $X$ from $q$ to $t_\epsilon$, the last point on $X$ in the assigned order. Then $Y_\epsilon$ is a finite polygonal arc and by elementary means (see Graueb [6]) can be straightened out to a segment $Y'_\epsilon$ leaving a neighborhood of $p$ pointwise fixed. Denote this semilinear homeomorphism by $f_\epsilon$. Let a neighborhood of $p$ that is pointwise fixed by $f_\epsilon$ be $Z_\epsilon$. We note $X$ becomes $f_\epsilon(X)$ and the part of $f_\epsilon(X)$ exterior to $F(p, \epsilon)$ is a segment and hence lies in a plane. If $\epsilon' > \epsilon$, $F(p, \epsilon')$ meets $f_\epsilon(X)$ in at most one point. Then $K = f_\epsilon^{-1}[F(p, \epsilon')]$ is a topological 2-sphere that meets $X$ in a single point if $\epsilon' - \epsilon$ is sufficiently small. Since the points of $Z_\epsilon$ have remained pointwise fixed, $X$ is LAU at $p$.

**Remark 1.** The segment $Y'_\epsilon$ referred to above may be taken to lie on a line through $p$.

**Remark 2.** Among the arcs locally tame modulo an end-point, the LAU condition is no further restriction of the embedding.

**Remark 3.** Let $X$ be a union of two arcs $X'$ and $X''$ meeting at $p$. Assume $X' \setminus p$ and $X'' \setminus p$ are locally tame. If $X$ has penetration index = 2 at $p$, then $X'$ and $X''$ are each tame and $X$ is at most mildly wild [8]. Applying the above calculations to $X'$ and $X''$, we see that the hypothesis that $X$ be LAU at $p$ implies that the homeomorphisms used in straightening $X'$ and $X''$ be consistent, i.e. there is a single homeomorphism that straightens each of $X'$, $X''$ except for a neighborhood of $p$. This will be taken as our definition of LAU at an interior point of an arc below.

**Theorem 2.** Among the mildly wild arcs, LAU and WPU are equivalent properties.

**Proof.** Let $X = X' \cup X''$ be mildly wild, where $X' \cap X'' = \{p\}$, $X'$, $X''$ are
tame and $X$ is LAU at $p$. Given $\varepsilon > 0$, define $U = S(p, \varepsilon) \cap X$. Let $V$ be the component of $S(p, \varepsilon/3) \cap X$ determined by $p$. By the LAU property, there is a neighborhood $W$ of $p$ and a homeomorphism $\xi$ of $R^3$ on $R^3$ such that

(i) $\xi|W = \text{identity},$

(ii) $\xi(U \setminus V)$ is a subset of plane $\pi$. Diminishing $W$ if necessary we assume $W = S(p, \delta)$ and for all $x$ in $W$ the arc $xp$ has a diameter $< \varepsilon/3$. If $q_{-1}$ and $q_{+1}$ are the end-points of $X$, let $X$ be reparametrized so that $q_{-1}$ corresponds to $t = -1$, $p$ to $t = 0$ and $t = +1$ to $q_1$. Let $A$ denote the first component of $(q_{-1}p) \cap \pi$ with one boundary component on $F(p, \varepsilon')$, $\varepsilon' > \varepsilon$, and one on $F(p, \varepsilon)$. Let $B$ denote the last component of $(pq_1) \cap \pi$ with one boundary component on $F(p, \varepsilon)$ and one on $F(p, \varepsilon')$. All other components of $\pi \cap (X \setminus W)$ can be pushed into $S(p, \varepsilon) \setminus W$ without moving the points of $W$ by a homeomorphism $\xi_2$. The set $\{S(p, \varepsilon') \setminus S(p, \varepsilon)\} \cap \pi$ is an annulus with two components of

$\{\{S(p, \varepsilon') \setminus S(p, \varepsilon)\}\} \cap X,$

call them $A$ and $B$, stretching from $F(p, \varepsilon')$ to $F(p, \varepsilon)$. Let $H$ be a simple closed curve in this annulus piercing both of $A$ and $B$ just once. Then define $K = H \times [-\varepsilon, +\varepsilon] \cup (\text{Int } H \times \varepsilon) \cup \{(\text{Int } H) \times -\varepsilon\}$. Clearly $K$ is a topological 2-sphere containing $p$ in its interior and $K \cap X = a$ of pairs of points, one on $A$, one on $B$ and $\text{diam } K < 2\varepsilon' + 2\varepsilon < 4\varepsilon$ (using a rectangular metric).

By construction, $\text{diam } K < 3\varepsilon$ if $\varepsilon' - \varepsilon$ is sufficiently small. Taking $h = (\xi_2\xi)^{-1}$ we have

(0) $h(p) = p,$

(i) $\text{diam } K < 3\varepsilon,$

(ii) $\text{card } K \cap h^{-1}(X) = 2$, i.e., $X$ is WPU at $p$.

In the converse direction, suppose $X = A \cup B$, where $A$ and $B$ are tame and $A \cap B = \{p\}$ is a common end-point. Assuming $X$ is WPU at $p$ we want to prove $X$ is LAU at $p$. Given $\varepsilon > 0$, there is a topological 2-sphere $K$ containing $p$ in its interior and a homeomorphism $h$ of $R^3$ on $R^3$ such that

(0) $h(p) = p,$

(i) $\text{diam } h^{-1}(K) < \varepsilon,$

(ii) $K \cap X = a \cup b$, $a \in A$ and $b \in B$.

Let $A_t$ denote the component of $A' \setminus a$ not containing $p$ and $B_t$ the component of $B' \setminus b$ not containing $p$. Then $A_t$ and $B_t$ are tame disjoint arcs defined by the WPU property. By the smoothing techniques of [1] or [9] we may choose $A_1$ and $B_1$ as polygonal arcs. Again, by elementary techniques $A_1 \cup B_1$ may be flattened out into a plane leaving a neighborhood $S$ of $p$ pointwise fixed by a homeomorphism $g$.

If $U$ is any neighborhood of $p$ large enough to contain $X \setminus g(A_1 \cup B_1)$, then given any $V(p) \subset U$ there is a $W(p) \subset V \subset S$ so that

(i) $g|W = \text{identity},$

(ii) $g|U \setminus V$ is a subset of a plane.

This serves in the definition of LAU for $X$ at $p$.

**Theorem 3.** Let $\Gamma$ again denote “The remarkable simple closed curve.” Then $\Gamma$ fails to be either locally unknotted (LU) or locally peripherally unknotted (LPU) at $p$. License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Proof. Suppose \( \Gamma \) is LU at \( p \). Then there is a disk \( D \) containing a neighborhood of \( p \) in \( \Gamma \). Since \( \Gamma \) is locally tame mod \( p \), there is no loss in choosing \( D \) locally polyhedral mod \( p \). Let \( Y = A \cup B \) be a neighborhood of \( p \) in \( \Gamma \) where \( A \) is a straight line interval with \( p \) as an end-point and \( B \) the closure of the complement in \( Y \). Then \( A \) and \( B \) are equivalently embedded in \( \mathbb{R}^3 \) by Theorem 5 of [2]. Hence \( A \) and \( B \) are both tame. The existence of \( D \) means \( A \cup B \) has the A1P at \( p \). Thus \( A \cup B \) is tame by [8], a contradiction. Hence \( \Gamma \) is locally knotted at \( p \) (i.e. no such \( D \) exists).

If \( \Gamma \) were LPU at \( p \), \( \Gamma \) would be expressible as a union of two nonoverlapping arcs \( qap, qbp \) denoted by \( A, B \), respectively. At least one of \( A, B \) is tame, say \( A \). There is a disk \( D \) whose boundary contains \( A \) and \( D \) may be chosen tame. By choosing \( D \) carefully we can arrange that \( D \cap B = \{ p \} \). Since \( \Gamma \) is LPU at \( p \), \( B \) is LPU at \( p \). Hence \( B \) is tame. It follows that \( A \cup B \) is part of a tame arc and hence tame, a contradiction.

Thus, the embedding of \( \Gamma \) at \( p \) is more complicated than that of either 1.2 or 1.4 of [3]. \( \Gamma \) is not a union of two tame nonoverlapping arcs but is a countable union of such tame nonoverlapping arcs.

References

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