CARTAN MATRICES, FINITE GROUPS OF QUATERNIONS, AND KLEINIAN SINGULARITIES

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To H.S.M. Coxeter for his 70th birthday

Abstract. The eigenvectors of the Cartan matrices of affine type $\tilde{A}_r$, $\tilde{D}_r$, $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$ can be taken to be the columns of the character tables of the finite groups of quaternions.

Proposition 1. The Cartan matrices of affine type $\tilde{A}_r$, $\tilde{D}_r$, $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$ are positive semidefinite. Their eigenvectors can be taken to be the columns of the character table of the corresponding finite group of quaternions [3, Chapters 6, 7], namely the cyclic group $\mathbb{Z}_{r+1}$, the binary dihedral (or dicyclic) group of order $4r - 8$, the binary tetrahedral, binary octahedral, and binary icosahedral group respectively.

We remark that any matrix with such eigenvectors necessarily commutes with the matrices of the regular representation of the representation algebra of the appropriate finite group $G$. Only for $\tilde{A}_0, \tilde{A}_1$ and $\tilde{E}_8$ are the eigenvalues simple and each eigenvector determined to within a scalar multiple.

For any (complex) representation $R$ of $G$, we construct a graph $\Gamma_R$ with the irreducible representations of $G$ as its nodes and $m_{jk}$ (possibly zero) directed edges from $R_j$ to $R_k$ where $R \otimes R_j = \bigoplus_k m_{jk} R_k$. We convene that an undirected edge between $R_j$ and $R_k$ represent the pair of directed edges from $R_j$ to $R_k$ and from $R_k$ to $R_j$.

Proposition 2. Each of the five types affine group described above has a faithful two-dimensional representation $R_Q$ such that $\Gamma_{R_Q}$ is the Coxeter graph of the corresponding affine type.

Taking $R = R_Q$ we deduce that the Cartan matrix, $C$, of $\Gamma_{R_Q}$ satisfies $C = 2I - M$ where $M = M_{R_Q} = (m_{jk})$. Proposition 1 follows as does the fact that the eigenvalues of $M$ are the character values afforded by $R_Q$.

Singularities on algebraic varieties were studied by Schläfli in 1863 and by Cayley in 1869 although it seems that Du Val [4] was the first to relate the singularities to finite groups. His book [5, §5.40] contains a description of this relationship in terms of the topology of the spherical neighbourhood of the
singularity. Steinberg [8, p. 156] writes: "Each singularity is realized naturally in the corresponding algebraic group, via a 'ridge' of singularities on the unipotent variety along its subregular subvariety." Brieskorn [1] remarks on the connection with finite groups which is discussed from an algebraic point of view in the forthcoming book of Slodowy [7].

Added in proof. The connected undirected graphs with adjacency matrix having maximum eigenvalue 2 are precisely the Coxeter graphs above, together with the graph $A_\infty$ which is the representation graph for $SU_2$ with $R = R_Q$, its natural two-dimensional representation. From this graph one may obtain each of the finite graphs as embeddings by restricting $R$ to a finite subgroup of $SU_2$. An interpretation of the dual of the $A_\infty$ graph as the Dynkin curve of an appropriate singularity is wanting.

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References