A VARIANT OF THE CHAIN RULE FOR DIFFERENTIAL CALCULUS

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Abstract. A version of the chain rule is developed which can be applied to the construction of solutions to quasi-linear hyperbolic partial differential equations.

In this note we present a variant of the usual chain rule for differential calculus which is extremely useful in the demonstration (see [3]) that certain types of quasi-linear unbounded vector fields generate continuous flows. We consider a composition $f \circ \alpha$, where $f$ maps an open subset of a Banach space $Y$ to a Banach space $Z$, and $\alpha$ is a curve whose image is contained in the domain of $f$. By strengthening the assumptions on the differentiability of $f$, we can weaken the assumptions on $\alpha$ to something less than continuity in $Y$.

We make the following assumptions throughout this article: $X$, $Y$, and $Z$ are Banach spaces, with $Y$ continuously and densely included in $X$; $V$ is an open subset of $Y$, and $f : V \to Z$; $[a, b] \subset \mathbb{R}$, and $\alpha : [a, b] \to V$. $B(X, Z)$ will denote the space of continuous linear maps from $X$ to $Z$, $B(Y, Z)$ the space of continuous linear maps from $Y$ to $Z$. If $l \in B(Y, Z)$ has an extension to an element of $B(X, Z)$, then we will use the same symbol ($"l"$, in this case) to denote the extension, and $\|l\|_{X,Z}$ will denote the norm of the extension.

**Lemma.** Let $t \in [a, b]$, and assume that $\alpha : [a, b] \to X$ is differentiable at $t$. Assume that $f$ has a Gateaux derivative at $\alpha(t)$, and that $Df(\alpha(t))$ extends to an element of $B(X, Z)$. Assume in addition that there exists $k > 0$ such that $\|f(v_2) - f(v_1)\|_Z < k\|v_2 - v_1\|_X$ for each $v_1, v_2 \in V$. Then $f \circ \alpha$ is differentiable at $t$, and $(f \circ \alpha)'(t) = Df(\alpha(t))(\alpha'(t))$.

**Proof.** Since $Y$ is dense in $X$, there exists a sequence $\{y_n\}_{n \in \mathbb{N}}$ of elements of $Y$ which converges in $X$ to $\alpha'(t)$. For each $n \in \mathbb{N}$ and $h \neq 0$,

$$\left\|h^{-1}\left[f(\alpha(t + h)) - f(\alpha(t))\right] - Df(\alpha(t))(\alpha'(t))\right\|_Z$$

$$< \left\|h^{-1}\left[f(\alpha(t + h)) - f(\alpha(t) + hy_n)\right]\right\|_Z$$

$$+ \left\|h^{-1}\left[f(\alpha(t) + hy_n) - f(\alpha(t))\right] - Df(\alpha(t))(y_n)\right\|_Z$$

$$+ \|Df(\alpha(t))(y_n - \alpha'(t))\|_Z.$$
Now,
\[
\|h^{-1}\left[f(\alpha(t + h)) - f(\alpha(t))\right]\|_Z = |h|^{-1}\|f(\alpha(t + h)) - f(\alpha(t)) - hy_n\|_X
\]
\[
< k|h|^{-1}\|\alpha(t + h) - \alpha(t) - hy_n\|_X
\]
\[
< k|h|^{-1}\|\alpha(t + h) - \alpha(t) - ha'(t)\|_X + k\|\alpha'(t) - y_n\|_X,
\]
and
\[
\|Df(\alpha(t))(\alpha'(t))\|_Z < \|Df(\alpha(t))\|_{X,Z}\|y_n - \alpha'(t)\|_X.
\]

Thus, for each \(n \in \mathbb{N}\),
\[
\limsup_{|h| \to 0} \|h^{-1}\left[f(\alpha(t + h)) - f(\alpha(t))\right] - Df(\alpha(t))(\alpha'(t))\|_Z
\]
\[
< (k + \|Df(\alpha(t))\|_{X,Z})\|y_n - \alpha'(t)\|_X.
\]

Since \(\|y_n - \alpha'(t)\|_X \to 0\) as \(n \to \infty\), the lemma is proved. □

**Theorem.** Assume that \(V\) is convex, that \(f\) has a Gateaux derivative at each point of \(V\), and that each \(Df(\alpha)\) has an extension to an element of \(B(X, Z)\), and that \(Df(V)\) is a bounded subset of \(B(X, Z)\). Assume in addition that \(\alpha(\cdot)\) is absolutely continuous and differentiable almost everywhere from \([a, b]\) to \(X\). Then \(f \circ \alpha\) is absolutely continuous and differentiable almost everywhere from \([a, b]\) to \(Z\), and \((f \circ \alpha)'(t) = Df(\alpha(t))(\alpha'(t))\) for each \(t\) at which \(\alpha'(t)\) exists.

**Proof.** Choose \(c > 0\) such that \(\|Df(\alpha)\|_{X,Z} < c\) for every \(\alpha \in V\). From the Mean Value Theorem it follows that \(\|f(v_2) - f(v_1)\|_Z < c\|v_2 - v_1\|_X\) for each \(v_1, v_2 \in V\). The above lemma then implies that \((f \circ \alpha)'(t) = Df(\alpha(t))(\alpha'(t))\) for each \(t\) at which \(\alpha'(t)\) exists. Thus, \(f \circ \alpha\) is differentiable almost everywhere. Since \(\alpha(\cdot)\) is absolutely continuous from \([a, b]\) to \(X\), the estimate \(\|f(v_2) - f(v_1)\|_Z < c\|v_2 - v_1\|_X\) for every \(v_1, v_2 \in V\) implies that \(f \circ \alpha\) is absolutely continuous from \([a, b]\) to \(Z\). □

It is easy to use our lemma to produce versions of the above theorem in which the domain of \(\alpha\) is permitted to be an open subset of an arbitrary Banach space and \(\alpha\) is assumed to be Gateaux differentiable when regarded as a map into \(X\) (cf. the chain rule for \(\beta\)-differentiability in [1, §5]). However, many such generalizations are possible and so, having no specific application in mind for such a generalization, we have chosen to restrict our attention in this note to a version of demonstrated usefulness.

The proof of our lemma is an adaptation of the proof of a weaker result which appears in [2].

**REFERENCES**


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