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SCHUR INDICES AND THE GALOIS GROUP

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ABSTRACT. In this note we show that the order of the Schur index of an irreducible representation divides the order of a certain subgroup of the Galois group of a cyclotomic extension of the ground field.

For F a field, G a finite group of exponent n and χ an irreducible character of G, we let $m_F(\chi)$ denote the Schur index of χ over F. A famous theorem of Brauer states that $m_F(\chi) = 1$ if $\zeta_n \in F$ where ζ_n denotes a primitive nth root of unity. In 1975, Goldschmidt and Isaacs [3] showed that $p \nmid m_F(\chi)$ if p is a prime with the property that the p-Sylow subgroup of $\mathrm{Gal}(F(\zeta_n)|F)$ is cyclic, except possibly when $p = 2$ and $\sqrt{-1} \notin F$. Fein [2] showed that the exceptional case of the previous theorem cannot hold if -1 is a sum of two squares in F.

In this note we strengthen the previous results by showing

THEOREM. *Let F be a field of characteristic* 0, *n a positive integer and* $\mathrm{Gal}(F(\zeta_n)|F) \simeq H_1 \times H_2 \times \cdots \times H_k$ *with H_i cyclic and* $|H_i| \mid |H_{i-1}|$, $i = 2, \ldots, k$. *Let G be a finite group of exponent n and χ an irreducible character of G. Then*

$$m_F(\chi) \mid |H_2| \quad \textit{if } m_F(\chi) \not\equiv 2 \pmod 4$$

and

$$m_F(\chi) \mid 2|H_2| \quad \textit{if } m_F(\chi) \not\equiv 2 \pmod 4.$$

Before proving the theorem, we need an elementary lemma.

LEMMA. *Suppose $T = T_1 \times T_2 \times \cdots \times T_k$ with T_i finite cyclic groups and $|T_i| \mid |T_{i-1}|$, $i = 2, \ldots, k$. Let $a, b \in T$ each be of order r with $\langle a \rangle \cap \langle b \rangle = \{e\}$ ($\langle a \rangle$ is the group generated by a). Then $r \mid |T_2|$.*

PROOF. Write $H = T_2 \times \cdots \times T_k$. Then $|T_2|$ is the exponent of H. Write $a = (t_1, h_1)$, $b = (t_2, h_2)$ with $t_i \in T_1$, $h_i \in H$. Let $s_i = |\langle h_i \rangle|$, $i = 1, 2$. Then $s_i | r$ and $a^{s_1} = (t_1^{s_1}, e)$, $b^{s_2} = (t_2^{s_2}, e)$. As $\langle a \rangle \cap \langle b \rangle = \{e\}$, $(|\langle a^{s_1} \rangle|, |\langle b^{s_2} \rangle|) = 1$ and $(r/s_1, r/s_2) = 1$. Thus $(s_2 r, s_1 r) = s_1 s_2$. But $(s_2 r, s_1 r) = r(s_1, s_2)$, so $r = s_1 s_2/(s_1, s_2)$ and r is the least common multiple of s_1 and s_2. Hence H contains an element of order r and $r \mid |T_2|$.

PROOF OF THEOREM. Let $L = Q(\zeta_n) \cap F$. Then L is a finite extension of Q and $\mathrm{Gal}(L(\zeta_n)|L) \simeq \mathrm{Gal}(F(\zeta_n)|F)$. But F an extension of L implies $m_F(\chi) | m_L(\chi)$ and so it is sufficient to show that $m_L(\chi) \mid |H_2|$, and we henceforth suppose that F is a finite extension of Q.

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Suppose now that $q^s \| m_F(\chi)$ with q a prime. Then there is a rational prime p and a prime P lying over p in F so that localization at P gives $q^s \| m_{F_P}(\chi)$. Assume that $q^s \neq 2$ or $p \neq 2$.

Let ϕ: $\mathrm{Gal}(F_P(\zeta_n)|F_P) \to \mathrm{Gal}(F(\zeta_n)|F)$ given by $\phi(\sigma) = \sigma|_{F(\zeta_n)}$ for $\sigma \in \mathrm{Gal}(F_P(\zeta_n)|F_P)$, ϕ is an injection. Write $n = p^e t$ with $(t, p) = 1$. Then $F_P(\zeta_t)$ is a cyclic extension of F_P since it is a totally unramified extension. If $p = 2$, the Schur subgroup of F_P has at most two elements [5, Theorem 5.5] and so $q^s \neq 2$ or $p \neq 2$ implies $p \neq 2$ and $F_P(\zeta_p e)$ is a cyclic extension of F_P. Because $F_P(\zeta_n) = F_P(\zeta_p e, \zeta_t)$, $\mathrm{Gal}(F_P(\zeta_n)|F_P)$ is generated by two elements and thus so is the subgroup $\mathrm{Gal}(F_P(\zeta_n)|L)$ where $L = F_P(\chi) \subset F_P(\zeta_n)$. We can now write $\mathrm{Gal}(F_P(\zeta_n)|L)$ as the internal direct product of two cyclic groups A_1 and A_2. Let S_i be the fixed field of A_i.

We now show that $q^s |\ |A_i|$ for $i = 1, 2$. If not, we can suppose without loss of generality, that $q^u \|\ |A_1|$ with $0 \leqslant u < s$.

Since $\deg(S_2|L) = A_1$, by Theorems 4.21 and 9.23 of Albert [1], $q^{s-u} \| m_{S_2}(\chi)$. But $S_2(\zeta_n) = F_P(\zeta_n)$ and $\mathrm{Gal}(S_2(\zeta_n)|S_2) \simeq A_2$ is cyclic. If q is odd, by the theorem of Goldschmidt and Isaacs [3], $q \nmid m_{S_2}(\chi)$ giving a contradiction. If $q = 2$ and $p \neq 2$ then -1 is a sum of two squares [4, Lemma 2.2], in $Q_P \subset F_P$ and so by the theorem of Fein [2], $q \nmid m_{S_2}(\chi)$ again giving our contradiction. Hence we can assume $q^s |\ |A_i|$ for $i = 1, 2$.

Let $\sigma_i \subset A_i$ for $i = 1, 2$ be elements of order q^s. $\langle \sigma_1 \rangle \cap \langle \sigma_2 \rangle = \{e\}$ since $A_1 \cap A_2 = \{e\}$. Then $\phi(\sigma_1)$ and $\phi(\sigma_2)$ are again elements in $\mathrm{Gal}(F(\zeta_n)|F)$ of order q^s with $\langle \phi(\sigma_1) \rangle \cap \langle \phi(\sigma_2) \rangle = \{e\}$. By the lemma $q^s |\ |H_2|$ and the theorem is established.

We remark that the proof shows that $q^s |\ |H_2|$ unless $q^s = 2$ and $p = 2$.

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