A NONLINEAR ERGODIC THEOREM
FOR AN AMENABLE SEMIGROUP
OF NONEXPANSIVE MAPPINGS IN A HILBERT SPACE

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Abstract. We prove a nonlinear ergodic theorem for noncommutative semigroups
of nonexpansive mappings in a Hilbert space. Furthermore, we give a necessary
and sufficient condition for a noncommutative semigroup to have a fixed point.

1. Introduction. Let $H$ be a real Hilbert space with norm $\| \cdot \|$ and inner product
$\langle \cdot, \cdot \rangle$ and $C$ a nonempty closed convex subset of $H$. A mapping $T: C \to C$ is
called nonexpansive on $C$, or $T \in \text{Cont}(C)$ if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for every } x, y \in C.$$

Let $F(T)$ be the set of fixed points of $T$, that is, $F(T) = \{z \in C: Tz = z\}$. Then,
the set $F(T)$ is obviously closed and convex. Let $S = \{S(t): t \geq 0\}$ be a family of
nonexpansive mappings of $C$ into itself such that $S(0) = I$, $S(t + s) = S(t)S(s)$ for
all $t, s \in [0, \infty)$ and $S(t)x$ is continuous in $t \in [0, \infty)$ for each $x \in C$. Then, $S$
is called a nonexpansive semigroup on $C$. The fixed point set $F(S)$ of $S$ is defined by

$$F(S) = \{x \in C: S(t)x = x \quad \text{for all } t \in [0, \infty)\}.$$

The first nonlinear ergodic theorem for nonexpansive mappings was established
by Baillon [1]: Let $C \subset H$, $T \in \text{Cont}(C)$ and $F(T) \neq \emptyset$. Then the Cesàro means

$$S_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} T_kx$$

converge weakly as $n \to +\infty$ to a fixed point of $T$ for each $x \in C$. A corresponding
result for nonexpansive semigroups on $C$ was given by Baillon [2] and
Baillon-Brézis [3]. Nonlinear ergodic theorems for general commutative semigroups
of nonexpansive mappings were given by Brézis-Browder [4] and Hirano-Takahashi
[6].

In this paper, we prove a nonlinear ergodic theorem for an amenable semigroup
of nonexpansive mappings of $C$ into itself. Furthermore we obtain a necessary and
sufficient condition for a left amenable semigroup of nonexpansive mappings to
have a common fixed point. This is a generalization of Pazy's results [7] and [9].

2. Nonlinear ergodic theorem. Let $S$ be an abstract semigroup and $m(S)$ the
Banach space of all bounded real valued functions on $S$ with the supremum norm.
For each $s \in S$ and $f \in m(S)$, we define elements $f_s$ and $f^s$ in $m(S)$ given by
Let $f(t) = f(st)$ and $f'(t) = f(ts)$ for all $t \in S$. An element $\mu \in m(S)^*$ (the dual space of $m(S)$) is called a mean on $S$ if $\|\mu\| = \mu(1) = 1$. A mean $\mu$ is called left (right) invariant if $\mu(f) = \mu(f)$ ($\mu(f') = \mu(f)$) for all $f \in m(S)$ and $s \in S$. An invariant mean is a left and right invariant mean. A semigroup which has a left (right) invariant mean is called left (right) amenable. A semigroup which has an invariant mean is called amenable. Day [5] proved that a commutative semigroup is amenable. We also know that $\mu \in m(S)^*$ is a mean on $S$ if and only if
\[
\inf\{f(s); s \in S\} < \mu(f) < \sup\{f(s); s \in S\}
\]
for every $f \in m(S)$.

Now we prove a nonlinear ergodic theorem for noncommutative semigroups of nonexpansive mappings in a Hilbert space. The proof employs the methods of [8], [11] and [12].

**Theorem 1.** Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ and $S$ be an amenable semigroup of nonexpansive mappings $t$ of $C$ into itself. Suppose
\[
F(S) = \bigcap \{F(t); t \in S\} = \emptyset.
\]
Then, there exists a nonexpansive retraction $P$ of $C$ onto $F(S)$ such that $Pt = tP = P$ for every $t \in S$ and $Px \in \text{co}\{tx; t \in S\}$ for every $x \in C$, where $\text{co} A$ is the closure of the convex hull of $A$.

**Proof.** Let $\mu$ be an invariant mean on $S$ and $x \in C$. Then, since $F(S) \neq \emptyset$, $\{tx; t \in S\}$ is bounded and hence, for each $y$ in $H$, the real-valued function $t \to \langle tx, y \rangle$ is in $m(S)$. Denote by $\mu_t(x, y)$ the value of $\mu$ at this function. By linearity of $\mu$ and of the inner product, this is linear in $y$; moreover, since
\[
|\mu_t(x, y)| \leq \|\mu\| \cdot \sup_{t} |\langle tx, y \rangle| \leq \left( \sup_{t} \|tx\| \right) \cdot \|y\|,
\]
it is continuous in $y$, so by the Riesz theorem, there exists an $x_0 \in H$ such that $\mu_t(x, y) = \langle x_0, y \rangle$ for every $y \in H$. Setting $Px = x_0$, we have $Px \in \text{co}\{tx; t \in S\}$.

In fact, if $Px \notin \text{co}\{tx; t \in S\}$, then by the separation theorem there exists a $y_0 \in H$ such that
\[
\langle Px, y_0 \rangle < \inf_{t} \langle tx, y_0 \rangle < \inf_{t} \langle z, y_0 \rangle; z \in \text{co}\{tx; t \in S\} \}.
\]
So, we have
\[
\inf_{t} \langle tx, y_0 \rangle < \mu_t(x, y_0) = \langle Px, y_0 \rangle < \inf_{t} \langle z, y_0 \rangle; z \in \text{co}\{tx; t \in S\} \} \}
\]
This is a contradiction. Let $s \in S$. Then we have
\[
0 \leq \|tx - x_0\|^2 - \|sx_0 - x_0\|^2 \\
\leq \|tx - sx_0\|^2 + 2\langle tx - sx_0, sx_0 - x_0 \rangle \\
\leq \|sx_0 - x_0\|^2 - \|tx - sx_0\|^2
\]
and hence

\begin{align*}
0 & < \mu \left( \|tx - sx_0\|^2 + 2\langle tx - sx_0, sx_0 - x_0 \rangle + \|sx_0 - x_0\|^2 - \|sx_0 - x_0\|^2 \right) \\
& = \mu \|tx - sx_0\|^2 + 2\langle x_0 - sx_0, sx_0 - x_0 \rangle \\
& + \|sx_0 - x_0\|^2 - \mu \|tx - sx_0\|^2 \\
& = 2\langle x_0 - sx_0, sx_0 - x_0 \rangle + \|sx_0 - x_0\|^2 \\
& = -\|x_0 - sx_0\|^2.
\end{align*}

This implies \(sx_0 = x_0\) for every \(s \in S\) and hence we have \(sPx = Px\) for every \(s \in S\). From

\[\langle Psx, y \rangle = \mu \langle tsx, y \rangle = \mu \langle tx, y \rangle = \langle Px, y \rangle\]

and

\[\langle P^2x, y \rangle = \mu \langle tPx, y \rangle = \mu \langle Px, y \rangle = \langle Px, y \rangle,\]

it follows that \(Ps = P\) for every \(s \in S\) and \(P^2 = P\). At last, we prove that \(P\) is nonexpansive. In fact, we have

\[\|Px - Py\|^2 = \langle Px - Py, Px - Py \rangle = \mu \langle tx - ty, Px - Py \rangle \leq \left( \sup_{t} \|tx - ty\| \right) \cdot \|Px - Py\| \leq \|x - y\| \cdot \|Px - Py\|\]

for every \(x, y \in C\).

As a direct consequence, we have

**Corollary 1.** Let \(C\) be a nonempty closed convex subset of a real Hilbert space \(H\) and \(S\) be a commutative semigroup of nonexpansive mappings \(t\) of \(C\) into itself. Suppose that \(F(S) \neq \emptyset\). Then there exists a nonexpansive retraction \(P\) of \(C\) onto \(F(S)\) such that \(Pt = tP = P\) for every \(t \in S\) and \(Px \in \text{co}\{tx: t \in S\}\) for every \(x \in C\).

By the method of Theorem 1, we can prove the following

**Theorem 2.** Let \(C\) be a nonempty closed convex subset of a real Hilbert space \(H\) and \(S\) be a left amenable semigroup of nonexpansive mappings \(t\) of \(C\) into itself. Then, \(F(S) \neq \emptyset\) if and only if there exists an \(x_0 \in C\) such that \(\{tx_0: t \in S\}\) is bounded.

As direct consequences, we obtain Pazy's results [7] and [9].

**Corollary 2.** Let \(C\) be a nonempty closed convex subset of a real Hilbert space \(H\) and \(T\) be a nonexpansive mapping of \(C\) into itself. Then, \(F(T) \neq \emptyset\) if and only if there exists an element \(x_0 \in C\) such that the sequence \(\{T^nx_0: n = 1, 2, \ldots\}\) is bounded.

**Corollary 3.** Let \(C\) be a nonempty closed convex subset of a real Hilbert space \(H\) and \(S = \{S(t): t \geq 0\}\) be a nonexpansive semigroup on \(C\). Then, \(F(S) \neq \emptyset\) if and only if there exists an element \(x_0 \in C\) such that \(\{S(t)x_0: t \geq 0\}\) is bounded.
References


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