ON THE PRODUCT OF A RIESZ SET AND A SMALL $p$ SET

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Abstract. Let $Z^+$ be the semigroup consisting of all nonnegative integers. By a famous theorem of Bochner, $Z^+ \times Z^+$ is a Riesz set in $Z \otimes Z$. In this paper, we prove that the product set of a Riesz set and a small $p$ set is a small $p$ set.

1. Introduction. Let $T$ be the circle group and $Z^+$ the semigroup consisting of nonnegative integers. Then, by the famous F. and M. Riesz theorem, each measure on $T$ whose Fourier-Stieltjes transform vanishes off $Z^+$ is absolutely continuous with respect to the Lebesgue measure on $T$. Moreover, by a well-known theorem of Bochner, each measure on $T^2$ whose Fourier-Stieltjes transform vanishes off $Z^+ \times Z^+$ is absolutely continuous with respect to the Lebesgue measure on $T^2$.

In this paper, we prove that the product set of a Riesz set and a small $p$ set is a small $p$ set. We use Glicksberg's ideas [1] and the theory of disintegration.

For a LCA group $G$, $C_c(G)$, $C_0(G)$, $L^1(G)$ and $M(G)$ denote the usual spaces. For a subset $E$ of $\hat{G}$, $M_E(G)$ denotes the space consisting of all measures in $M(G)$ whose Fourier-Stieltjes transforms vanish off $E$. We denote the Haar measure on $G$ by $m_G$.

Definition. Let $G$ be a LCA group. For a positive integer $p$, a closed subset $E$ of $G$ is called a small $p$ set if the following is satisfied:

For each $\mu \in M_E(G)$, $\mu^p (= \mu \ast \mu \ast \cdots \ast \mu$ ($p$ times)) belongs to $L^1(G)$. In particular, a small 1 set is called a Riesz set.

Lemma 1 [4]. Let $p$ be a positive integer. Then we have

$$t_1 t_2 \cdots t_p = \sum_{i=1}^p A_i \Phi_i(t_1, t_2, \ldots, t_p)^p$$

for each $(t_1, t_2, \ldots, t_p) \in C^p$,

where $A_i \in C$ (complex numbers) and $\Phi_i$ are linear forms of $t_1, t_2, \ldots, t_p$.

2. Main theorem.

Theorem 1. Let $G_1$ and $G_2$ be metrizable $\sigma$-compact LCA groups. Let $E_1$ be a small $p$ set in $\hat{G}_1$ and $E_2$ a Riesz set in $\hat{G}_2$. Then $E_1 \times E_2$ is a small $p$ set in $\hat{G}_1 \oplus \hat{G}_2$.

Proof. Let $\mu$ be a measure in $M_{E_1 \times E_2}(G_1 \oplus G_2)$. Let $\pi$ be the projection from $G_1 \oplus G_2$ onto $G_2$. Put $\eta = \pi(\mu)$ (continuous image under $\pi$). Then, by disintegration theory, there exists a family $\{\lambda_h\}_{h \in G_1}$ in $M(G_1 \oplus G_2)$ with the following properties:

Received by the editors November 27, 1979.


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0002-9939/81/0000-0075/$02.50

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(1) \( h \mapsto \lambda_h(f) \) is a Borel measurable function of \( h \) for each bounded Borel measurable function \( f \) on \( G_1 \oplus G_2 \),
(2) \( \text{supp}(\lambda_h) \subset \sigma^{-1}(\{h\}) = G_1 \times \{h\} \),
(3) \( \|\lambda_h\| < 1 \),
(4) \( \mu(g) = \int_{G_2} \lambda_h(g) \, d\eta(h) \) for each bounded Borel measurable function \( g \) on \( G_1 \oplus G_2 \).

By (2), we have \( d\lambda_h(x,y) = d\nu_h(x) \times d\delta_h(y) \), where \( \nu_h \in M(G_1) \) and \( \delta_h \) is the Dirac measure at \( h \). By the method used in [1, pp. 425–426], we have \( \nu_h \in M_{E_1}(G_1) \) a.e. \( h(\eta) \).

That is, there exists a Borel measurable set \( K \) in \( G_2 \) with \( \eta(G_2 \setminus K) = 0 \) such that \( \nu_h \in M_{E_1}(G_1) \) for \( h \in K \).

Claim 1. \( \eta \) belongs to \( L^1(G_2) \).
For \( \gamma_0 = (\gamma_1, \gamma_2) \in G_1 \oplus G_2 \), we have \( \sigma(\gamma_0, \mu) = \tilde{\mu}(-\gamma_1, \gamma - \gamma_2) \). Hence \( \sigma(\gamma_0, \mu) \) belongs to \( M_{E_1 + E_2}(G_2) \). Since \( E_2 \) is a Riesz set, \( E_2 + E_2 \) is also a Riesz set. Hence \( \sigma(\gamma_0, \mu) \) belongs to \( L^1(G_2) \). On the other hand, there exists a sequence \( \{ \rho_n \} \) in \( \text{Trig}(G_1 \oplus G_2) \) such that \( \lim_{n \to \infty} ||\rho_n - \mu|| = 0 \). Hence we have

\[
\lim_{n \to \infty} \| \sigma(\rho_n, \mu) - \sigma(\mu) \| = 0.
\]

Thus Claim 1 is proved.

Claim 2. \( (h_1, \ldots, h_p) \mapsto \lambda_{h_1} \ast \cdots \ast \lambda_{h_p}(g) = (\nu_{h_1} \ast \cdots \ast \nu_{h_p}) \times \delta_{\lambda_{h_1} \ast \cdots \ast \lambda_{h_p}}(g) \) is a Borel measurable function on \( G_2^p \) for each bounded Borel function \( g \) on \( G_1 \oplus G_2 \).

Let \( f \in C_c(G_1 \oplus G_2) \) and put \( f(x_1, \ldots, x_p) = f_1(x_1) \cdots f_p(x_p) \) for \( (x_1, \ldots, x_p) \in (G_1 \oplus G_2)^p \). Then \( (\lambda_{h_1} \ast \cdots \ast \lambda_{h_p})(f) = \lambda_{h_1}(f_1) \cdots \lambda_{h_p}(f_p) \) so that by (1)

\[
(5) (h_1, \ldots, h_p) \mapsto (\lambda_{h_1} \ast \cdots \ast \lambda_{h_p})(f) \text{ is Borel measurable.}
\]

Since \( \{ \Sigma f_1(x_1) \cdots f_p(x_p) : f_i \in C_c(G_1 \oplus G_2) \} \) is dense in \( C_0((G_1 \oplus G_2)^p) \), (5) is Borel measurable for \( f \in C_0((G_1 \oplus G_2)^p) \), hence, also for bounded Borel functions \( f \) on \( (G_1 \oplus G_2)^p \). Let \( \sigma_p(x_1, \ldots, x_p) = x_1 + \cdots + x_p \) for \( (x_1, \ldots, x_p) \in (G_1 \oplus G_2)^p \). Then for bounded Borel \( g \) on \( G_1 \oplus G_2 \) we have \( \lambda_{h_1} \ast \cdots \ast \lambda_{h_p}(g) = (\lambda_{h_1} \ast \cdots \ast \lambda_{h_p})(g \circ \sigma_p) \) and the claim follows.

Hence, by Claim 2, we can define a measure \( \xi \) in \( M(G_1 \oplus G_2) \) as follows

\[
\xi(f) = \int_{G_2} \int_{G_2} \cdots \int_{G_2} \left\{ (\nu_{h_1} \ast \nu_{h_2} \ast \cdots \ast \nu_{h_p}) \times \delta_{(\lambda_{h_1} \ast \cdots \ast \lambda_{h_p})}(f) \right\} \, \eta(h_1) \, \eta(h_2) \cdots \eta(h_p)
\]

for \( f \in C_0(G_1 \oplus G_2) \).

Claim 3. \( \xi = \mu^p \).

Let \( (\gamma_1, \gamma_2) \) be in \( G_1 \oplus G_2 \). Then we have

\[
\tilde{\xi}(\gamma_1, \gamma_2) = \prod_{i=1}^p \int_{G_2} \tilde{\nu}_h(\gamma_1)(-h_i, \gamma_2) \, d\eta(h_i)
\]

\[
= \{ \tilde{\mu}(\gamma_1, \gamma_2) \}^p = (\mu^p)(\gamma_1, \gamma_2).
\]

Thus we have \( \xi = \mu^p \).

Let \( E \) be a Borel measurable set in \( G_1 \oplus G_2 \) with \( m_{G_1 \oplus G_2}(E) = 0 \), where \( m_{G_1 \oplus G_2} \)
denotes the Haar measure on \( G_1 \oplus G_2 \). Then there exists a Borel measurable set \( F_2 \) in \( G_2 \) such that (i) \( m_{G_2}(F_2) = 0 \) and (ii) \( m_{G_1}(E_y) = 0 \) if \( y \notin F_2 \), where \( E_y = \{ x \in G_1; (x, y) \in E \} \). Let \( \alpha_p \) be the homomorphism from \( G_2^p \) onto \( G_2 \) such that \( \alpha_p(h_1, h_2, \ldots, h_p) = h_1 + h_2 + \cdots + h_p \). We note that \( \nu_{h_1} \ast \nu_{h_2} \ast \cdots \ast \nu_{h_p} \) belongs to \( L^1(G_1) \) for \( (h_1, h_2, \ldots, h_p) \in K^p \) by Lemma 1 and

\[
\eta^p(F_2) = (\eta \times \eta \times \cdots \times \eta)(\alpha_p^{-1}(F_2)).
\]

Hence, by Claim 1 and Claim 3, we have

\[
\begin{align*}
\mu^p(E) &= \int_{G_1} \cdots \int_{G_2} \left\{ (\nu_{h_1} \ast \nu_{h_2} \ast \cdots \ast \nu_{h_p}) \times \delta_{(h_1 + h_2 + \cdots + h_p)} \right\} \\
&= \int_{K^p} \left\{ \int_{G_2} \cdots \int_{G_2} (\eta \times \eta \times \cdots \times \eta)(h_1, h_2, \ldots, h_p) \right\} \\
&= \int_{K^p} \left\{ \int_{G_2} \cdots \int_{G_2} (\eta \times \eta \times \cdots \times \eta)(h_1, h_2, \ldots, h_p) \right\} \\
&\quad + \int_{K^p} \left\{ \int_{G_2} \cdots \int_{G_2} (\eta \times \eta \times \cdots \times \eta)(h_1, h_2, \ldots, h_p) \right\} = 0.
\end{align*}
\]

Hence we have \( \mu^p \in L^1(G_1 \oplus G_2) \). Thus \( E_1 \times E_2 \) is a small \( p \) set in \( G_1 \oplus G_2 \).

Q.E.D.

Dr. S. Saeki kindly pointed out the following Lemma 2 to the author.

**Lemma 2.** Suppose the product set \( F_1 \times E_2 \) of a small \( p \) set \( F_1 \) in \( \hat{G}_1 \) and a Riesz set \( E_2 \) in \( \hat{G}_2 \) is a small \( p \) set in \( \hat{G}_1 \oplus \hat{G}_2 \) for all metrizable LCA groups \( G_1 \) and \( G_2 \). Then the product set \( E_3 \times E_4 \) of a small \( p \) set \( E_3 \) in \( \hat{G}_3 \) and a Riesz set \( E_4 \) in \( \hat{G}_4 \) is a small \( p \) set in \( G_3 \oplus G_4 \) for all LCA groups \( G_3 \) and \( G_4 \).

**Proof.** Suppose there exists a measure \( \mu \) in \( M_{E_1 \times E_2}(G_3 \oplus G_4) \) such that \( \mu^p \) does not belong to \( L^1(G_3 \oplus G_4) \). Then, by Corollary 3 of [2], there exists a measure \( \sigma \) in \( M_\sigma(G_3 \oplus G_4) \) such that

\[
(\mu \ast \sigma)^p = \mu^p \ast \sigma^p \notin L^1(G_3 \oplus G_4).
\]

Since \( \sigma \in M_\sigma(G_3 \oplus G_4) \), there exist open \( \sigma \)-compact subgroups \( \Gamma_i \) of \( \hat{G}_i \) such that \( \text{supp}(\delta) \subseteq \Gamma_3 \times \Gamma_4 \) (\( i = 3, 4 \)). On the other hand, \( \hat{G}_3 \) and \( \hat{G}_4 \) are metrizable LCA groups. Evidently \( E_3 \cap \Gamma_3 \) is a small \( p \) set in \( \Gamma_3 \) and \( E_4 \cap \Gamma_4 \) is a Riesz set in \( \Gamma_4 \). Hence, by the hypothesis of this Lemma, \( (E_3 \cap \Gamma_3) \times (E_4 \cap \Gamma_4) \) is a small \( p \) set in \( \Gamma_3 \oplus \Gamma_4 \). Therefore, since \( \sigma \ast \mu \in M_{(E_3 \cap \Gamma_3) \times (E_4 \cap \Gamma_4)}(G_3 \oplus G_4) \), we have \( (\sigma \ast \mu)^p \in L^1(G_3 \oplus G_4) \).

This contradicts (A). Q.E.D.

**Lemma 3.** Let \( G_3 \) and \( G_4 \) be metrizable LCA groups. Let \( E_3 \) be a small \( p \) set in \( \hat{G}_3 \) and \( E_4 \) a Riesz set in \( \hat{G}_4 \). Then \( E_3 \times E_4 \) is a small \( p \) set in \( G_3 \oplus G_4 \).

**Proof.** Let \( \mu \) be a measure in \( M_{E_3 \times E_4}(G_3 \oplus G_4) \). Then there exist open metrizable \( \sigma \)-compact subgroups \( G_1 \subseteq G_3 \) and \( G_2 \subseteq G_4 \) such that \( \text{supp}(\mu) \) is contained in \( G_1 \oplus G_2 \). Let \( \pi_{G_2} \) be the projection from \( G_1 \oplus G_2 \) onto \( G_2 \). Put \( \eta' = \pi_{G_2}(|\mu|) \). Then,
by disintegration theory, there exists a family \( \{ \lambda_h \}_{h \in G_2} \) in \( M(G_1 \oplus G_2) \) such that

1. \( h \mapsto \lambda_h(f) \) is a Borel measurable function of \( h \) for each bounded Borel measurable function \( f \) on \( G_1 \oplus G_2 \),
2. \( \text{supp}(\lambda_h) \subset G_1 \times \{ h \} \),
3. \( ||\lambda_h|| < 1 \) and
4. \( \mu(g) = \int_{G_2} \lambda_h(g) \, d\eta(h) \) for each bounded Borel measurable function \( g \) on \( G_1 \oplus G_2 \).

Since \( G_1 \oplus G_2 \) is \( \sigma \)-compact and metrizable, there exists a countable dense set \( \mathfrak{C} = \{ f_m \} \) in \( C_0(G_1 \oplus G_2) \). For each \( g \in C_0(G_1 \oplus G_2) \), we define a function \( \hat{g} \) in \( C_0(G_3 \oplus G_4) \) by \( \hat{g}(x) = g(x) \) for \( x \in G_1 \oplus G_2 \) and \( \hat{g}(x) = 0 \) for \( x \notin G_1 \oplus G_2 \). We define measures \( \eta \in M(G_4) \) and \( \lambda_h \in M(G_3 \oplus G_4) \) \((h \in G_2)\) as follows

\[
\eta(F) = \eta'((G_2 \cap F) \text{ for a Borel measurable set } F \text{ in } G_4,
\lambda_h(F') = \begin{cases} 
\lambda_h(F' \cap G_1 \oplus G_2) & \text{if } h \in G_2, \\
0 & \text{if } h \in G_4 \setminus G_2,
\end{cases}
\]

for a Borel measurable set \( F' \) in \( G_3 \oplus G_4 \). Then we have the following

6. \( h \mapsto \lambda_h(f) \) is a Borel measurable function of \( h \) for each bounded Borel measurable function \( f \) on \( G_3 \oplus G_4 \),
7. \( \text{supp}(\lambda_h) \subset G_3 \times \{ h \} \subset G_3 \times \{ h \} \),
8. \( ||\lambda_h|| < 1 \),
9. \( \mu(g) = \int_{G_4} \lambda_h(g) \, d\eta(h) \) for each bounded Borel measurable function \( g \) on \( G_3 \oplus G_4 \).

From (7), we have \( d\lambda_h(x, y) = d\rho_h(x) \times d\delta_h(y) \), where \( \rho_h \) is a measure in \( M(G_3) \) with \( \text{supp}(\rho_h) \subset G_1 \). Noting that \( \text{supp}(\eta) \subset G_2 \), we may apply Lusin’s theorem and regularity of \( \eta \) to obtain for each positive integer \( n \) a compact subset \( K_n \) of \( \text{supp}(\eta) \) such that

(i) \( \eta(G_2 \setminus K_n) < 1/n \),
(ii) \( h \mapsto \lambda_h(f_m) \) is a continuous function on \( K_n \) for each \( f_m \in \mathfrak{C} = \{ f; f \in \mathfrak{C} \} \),
and
(iii) for each \( x \in K_n \) and neighborhood \( V \) of \( x \), \( \eta(V \cap K_n) > 0 \).
Since \( \mathfrak{C} \) is dense in \( C_0(G_1 \oplus G_2) = \{ f; f \in C_0(G_1 \oplus G_2) \} \) we may replace (ii) by

(ii)' \( h \mapsto \lambda_h(f) \) is continuous on \( K_n \) for each \( f \in C_0(G_1 \oplus G_2) \).

Claim 1. \( \rho_h \in M_c(G_3) \) for \( h \in K_n \).

Let \( f \in L^1(\hat{G}_3) \) with \( \text{supp}(f) \subset E_3^* \). Then since \( \hat{\mu}(\gamma_1, \gamma_2) = 0 \) for \( \gamma_1 \notin E_3 \) we have

\[
0 = \int_{\hat{G}_3} \hat{\mu}(\gamma_1, \gamma_2)f(\gamma_1) \, d\gamma_1
= \int_{\hat{G}_3} \int_{G_4} \int_{G_3} (-x, \gamma_1)|d\rho_h(x)|(-h, \gamma_2) \, d\eta(h)f(\gamma_1) \, d\gamma_1
= \int_{G_4} \int_{G_3} \hat{f}(x)|d\rho_h(x)|(-h, \gamma_2) \, d\eta(h)
= \int_{G_4} \rho_h(\hat{f})(-h, \gamma_2) \, d\eta(h).
\]
Hence, for each $F \in L^1(\mathbb{G}_4)$ and $f \in L^1(\mathbb{G}_2)$ with $\text{supp}(f) \subseteq E_3$ we have
\[
0 = \int_{\mathbb{G}_4} \int_{\mathbb{G}_4} v_h(f)(-h, \gamma_2) \, d\eta(h) \, F(\gamma_2) \, d\gamma_2
= \int_{\mathbb{G}_4} v_h(f) \hat{F}(h) \, d\eta(h). \tag{11}
\]

Since $L^1(\mathbb{G}_4)$ is dense in $C_0(\mathbb{G}_4)$, hence also in $L^1(\eta)$, (11) holds for all $F \in L^1(\eta)$. It follows from (6) that $h \mapsto v_h(f)$ is bounded and Borel measurable, hence in $L^\infty(\eta)$, and so by (11)
\[
v_h(f) = 0, \quad \eta\text{-a.e.} \tag{12}
\]

Let $\beta \in C_c(\mathbb{G}_4)$ satisfy $\beta = 1$ on $K_n$, $\beta = 0$ off $G_2$, and set $g(x, y) = (f \ast m_{G_4})(x) \beta(y)$, where $G_4$ is the annihilator of $G_1$. Then $g \in C_0(\mathbb{G}_1 \oplus \mathbb{G}_2)$ and, since $\text{supp}(v_h) \subseteq G_1$, we have $\lambda_n(g) = \nu_h((f \ast m_{G_1}) \beta) = v_h(f)$ for each $h \in K_n$. Hence, $h \mapsto v_h(f)$ is continuous on $K_n$ by (ii), and this together with (12) and (iii) shows that $v_h(f) = 0$ for each $h \in K_n$. Thus, for $h \in K_n$, $0 = v_h(f) = \int_{\mathbb{G}_4} v_h(\gamma_1) f(\gamma_1) \, d\gamma_1$.

Since $f$ is any function in $L^1(\mathbb{G}_3)$, $\text{supp}(f) \subseteq E_3$ we have $v_h(\gamma_1) = 0$ on $E_3$, and the claim is established. Moreover, since $\eta(G_3 \cup \bigcup_{K_n} K_n) = 0$ we have proved that $v_h \in M_{E_3}(G_3)$ a.a. $\eta(\eta)$.

Since $E$ is a Riesz set, we may prove that $\eta \in L^1(\mathbb{G}_4)$ by arguing as in Theorem 1.

**Claim 2.** $(h_1, h_2, \ldots, h_p) \mapsto \{(v_{h_1} \ast v_{h_2} \ast \cdots \ast v_{h_p}) \times \delta_{(h_1 + h_2 + \cdots + h_p)})(g) \}$ is a Borel measurable function on $(\mathbb{G}_4)^p$ for each $g \in C_0(\mathbb{G}_3 \oplus \mathbb{G}_4)$.

Indeed, since $v_h = 0$ if $h \not\in G_2$, we have $(v_{h_1} \ast v_{h_2} \ast \cdots \ast v_{h_p}) \times \delta_{(h_1 + h_2 + \cdots + h_p)} = 0$ for $(h_1, h_2, \ldots, h_p) \not\in (G_2)^p$.

On the other hand, $\lambda_n = v_h \times \delta_h$ may be regarded as a measure in $M(G_1 \oplus G_2)$.

Since $G_1$ and $G_2$ are $\sigma$-compact metrizable LCA groups, we may prove Claim 2 by arguing as in the proof of Claim 2, Theorem 1.

We now define a measure $\xi$ in $M(G_3 \oplus \mathbb{G}_4)$ as follows
\[
\xi(f) = \int_{G_4} \cdots \int_{G_4} \{(v_{h_1} \ast \cdots \ast v_{h_p}) \times \delta_{(h_1 + \cdots + h_p)}\}(f) \, d\eta(h_1) \cdots d\eta(h_p)
\]
for $f \in C_0(G_3 \oplus \mathbb{G}_4)$.

Then we have $\xi = \mu^p$. Let $E_0$ be a Borel measurable set in $G_3 \oplus \mathbb{G}_4$ with $m_{G_1 \oplus G_2}(E_0) = 0$. Put $E = E_0 \cap G_1 \oplus G_2$. Since $\text{supp}(\mu^p) \subseteq G_1 \oplus G_2$, we have $\mu^p(E_0) = \mu^p(E)$. Moreover, $m_{G_1 \oplus G_2}(E) = 0$. Thus we can prove that $\mu^p(E_0) = 0$ by using the techniques employed in Theorem 1. That is $\mu^p \in L^1(G_3 \oplus \mathbb{G}_4)$. This completes the proof.

From Lemma 2 and Lemma 3, we obtain the following main theorem.

**Theorem 2.** Let $G_1$ and $G_2$ be LCA groups. Let $E_1$ be a small $p$ set in $\mathbb{G}_1$ and $E_2$ a Riesz set in $\mathbb{G}_2$. Then $E_1 \times E_2$ is a small $p$ set in $G_1 \oplus G_2$. 
Finally, the author wishes to express his thanks to Dr. S. Saeki for his valuable advice.

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