PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 2, February 1981

THE FIRST EIGENVALUE OF THE LAPLACIAN FOR PLANE DOMAINS

CHRISTOPHER B. CROKE[1]

ABSTRACT. We prove an improved lower bound for the first eigenvalue of the Laplacian of a connected plane domain in terms of its inradius and connectivity.

In this note we find a lower bound for the first eigenvalue, λ_1, of the Dirichlet problem for the Laplacian of a connected plane domain D in terms of its inradius ρ (the radius of the largest disk contained in D), and the connectivity k (i.e., the number of boundary components).

Hayman [3] was the first to prove an inequality of this type. He proved $\lambda_1(D) > 1/900\rho^2$ in the case $k = 1$ (i.e., D simply connected). Next, Osserman [4] showed that

$$\lambda_1(D) \geqslant 1/4\rho^2, \qquad k = 1, 2,$$
$$\lambda_1(D) \geqslant 1/k^2\rho^2, \qquad k \geqslant 2.$$

In that paper Osserman suggests that one might be able to get a bound of the form $\lambda_1(D) \geqslant c/k\rho^2$. Taylor [6] proved that such a bound exists, although no explicit constant c was given. Recently Cheng [2] has shown, using a completely different method, that c can be taken to be $1/(14000\pi)^2$.

In this note we show

$$\lambda_1(D) \geqslant 1/2k\rho^2 \quad \text{for } k \geqslant 2 \qquad \left(\text{i.e. } c = \tfrac{1}{2}\right).$$

The method used here is similar to the one used in [4]. The method is useful only for plane domains, whereas Taylor's, Hayman's, and Cheng's methods are useful in higher dimensions or for variable curvatures.

THEOREM. *Let D be a connected, k-connected, domain in the plane with inradius ρ. Let A represent the area of D and L represent the total boundary length. Then*

$$L/A \geqslant 1/\rho, \qquad k = 1, 2. \tag{1}$$
$$L/A \geqslant 2/(1+\sqrt{k-1})\rho \geqslant \sqrt{2}/\sqrt{k}\,\rho, \qquad k \geqslant 2. \tag{2}$$
$$\lambda_1(D) \geqslant 1/4\rho^2, \qquad k = 1, 2. \tag{3}$$
$$\lambda_1(D) \geqslant 1/(1+\sqrt{k-1})^2\rho^2 \geqslant 1/2k\rho^2, \qquad k \geqslant 2. \tag{4}$$

Received by the editors March 24, 1980.
1980 *Mathematics Subject Classification*. Primary 52A40.
[1]Supported in part by NSF Grant MCS 76-01692.

0002-9939/81/0000-0080/$01.50

PROOF. Inequalities (1) and (2) (applied to subdomains of D) imply inequalities (3) and (4) by Cheeger's result [1] as modified by Osserman [4]. Inequality (1) (and (3)) were proved in [4]. It is a simple computation to see $\sqrt{2}\sqrt{k} \geqslant (1+\sqrt{k-1})$. Hence we need only show that for $k \geqslant 2$, $L/A \geqslant 2/(1+\sqrt{k-1})\rho$.

We consider two cases.

Case 1. $A \leqslant (1+\sqrt{k-1})^2\pi\rho^2$.

By the isoperimetric inequality for plane domains we have $L^2/A \geqslant 4\pi$. Hence $L/A \geqslant 2\sqrt{\pi}/\sqrt{A} \geqslant 2/(1+\sqrt{k-1})\rho$.

Case 2. $A \geqslant (1+\sqrt{k-1})^2\pi\rho^2$.

By a standard argument in the plane [4, p. 548] one has

$$\frac{\rho L}{A} \geqslant 1 - \frac{\pi(k-2)\rho^2}{A} \geqslant 1 - \frac{(k-2)}{(1+\sqrt{k-1})^2} = \frac{2}{(1+\sqrt{k-1})}.$$

This proves the theorem.

REMARKS. Inequalities (1) and (2) are sharp for $k = 2$, with equality for a circular annulus. For $k = 1$ inequality (1) is strict but it is also the best possible, as was noted by Santaló [5, p. 155], as one sees by considering long thin rectangles. For $k \geqslant 3$ inequality (2) is strict and not the best possible. In this case one could ask for the best constants $C(k)$ such that $L/A \geqslant C(k)/\sqrt{k}\,\rho$. The theorem gives $C(k) > 2\sqrt{k}/(1+\sqrt{k-1})$, thus asymptotically $C(k) \geqslant 2$ (i.e. for every $\varepsilon > 0$ and for sufficiently large k, $C(k) \geqslant 2-\varepsilon$). By considering large disks with triangularly packed points removed, one can see that asymptotically $C(k) \leqslant 2\sqrt{2\pi/3\sqrt{3}}$. Thus the theorem gives an estimate which is sharp for $k = 2$ and close to the best asymptotically.

As for inequality (4), Osserman [4, p. 552] gives examples where $\lambda_1(D) < \pi^2/k\rho^2$; thus the inequality is not too far from the best possible.

REFERENCES

1. J. Cheeger, *A lower bound for the smallest eigenvalue of the Laplacian*, Problems in Analysis, A Symposium in Honor of S. Bochner, Princeton Univ. Press, Princeton, N. J., 1970, pp. 195–199.

2. S.-Y. Cheng, *On the Hayman-Osserman-Taylor inequality* (preprint).

3. W. Hayman, *Some bounds for principal frequency*, Applicable Anal. **7** (1978), 247–254.

4. R. Osserman, *A note on Hayman's theorem on the bass note of a drum*, Comment. Math. Helv. **52** (1977), 545–555.

5. L. A. Santaló, *Sobre el circulo de radio maximo contenido en un recinto*, Rev. Un. Mat. Argentina **10** (1945), 155–167.

6. M. Taylor, *Estimate on the fundamental frequency of a drum*, Duke Math. J. **46** (1979), 447–453.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA 94720

Current address: Department of Mathematics, University of Pennsylvania, Philadelphia, Pennsylvania 19104