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CHARACTERIZATION OF THE SET-THEORETICAL GEOMETRIC REALIZATION IN THE NONEUCLIDEAN CASE

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ABSTRACT. A helpful feature in Milnor's geometric realization [1] of a simplicial set X is that each equivalence class admits one and only one element (optimal pair) of the form $(\bar{x}, \bar{t})$ where $\bar{x}$ is nondegenerate and $\bar{t}$ is an interior point. This realization and several other aspects of Algebraic Topology admit generalizations (R. Ruiz [3]) changing the cosimplicial topological space of the Δ^n's by a general one, say Y. This paper is devoted to establishing conditions on Y which guarantee the existence of such pairs on $R_Y(X)$ for every simplicial set X. (R_Y denotes the new realization via Y.)

0. Introduction. Let Y: $\Delta \to \mathrm{Top}$ be a cosimplicial topological space (a model of Top). It is well known that Y defines a covariant functor S_Y: $\mathrm{Top} \to \Delta^* S$ ($\Delta^* S$ is the category of simplicial sets) which is the generalization of the singular functor Sing: $\mathrm{Top} \to \Delta^* S$. S_Y admits a left adjoint R_Y, which generalizes Milnor's Geometric Realization [1], whose systematic use was exhibited in [3].

The structure of the geometric realization is fairly well known. In fact if we denote by $\underline{\Delta}$ the model of the topological Δ^n's, we get a set-theoretical model $V \cdot \underline{\Delta}$: $\Delta \to \mathrm{Top} \to S$ (V is the forgetful functor) and a commutative diagram

$$\begin{tikzcd} \Delta^* S \arrow[rr, "\|"] \arrow[dr, "R_{V\cdot\underline{\Delta}}"'] & & \mathrm{Top} \arrow[dl, "V"] \\ & S & \end{tikzcd}$$

where $R_{V\cdot\underline{\Delta}}(X) = \amalg X(n) \times V(\Delta^n)/\sim$ and $\sim$ is generated by the relation which identifies $(X(w)(x), t)$ with $(x, \underline{\Delta}(w)(t))$ for each w: $[n] \to [m]$, x in $X(m)$, $t \in \Delta^n$ and $n, m \geqslant 0$ (Milnor's relation). Each equivalence class has precisely one representative $(\bar{x}, \bar{t})$ where $\bar{x}$ is nondegenerate and $\bar{t}$ is an interior point. $R_{V\cdot\underline{\Delta}}(X)$ has already the set-theoretical features of a CW-complex with a copy of $V(\mathrm{int}\, \Delta^n)$ for each nondegenerate element $x \in X(n)$. To recover the topology of $|X|$ one gives to each Δ^n its topology of subspaces of R^{n+1} and to $R_{V\cdot\underline{\Delta}}(X)$ the weak topology over the cells induced by the Δ^n's.

For a general model Y: $\Delta \to \mathrm{Top}$ one still has the corresponding commutative diagram, i.e. $V \cdot R_Y = R_{VY}$, and $R_{VY}(X) = \amalg_n X(n) \times V(Y(n))/\sim$ as before. But the relation $\sim$ is intricate to look into and distinguished representatives of classes

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are in general missing which implies a damage in the (noneuclidean) cellular structure. The existence of those representatives is then crucial and strictly a set-theoretical matter. We devote this paper to the conditions on a model $Y\colon \Delta \to S$ in order for its realization to admit them. An example of uses of this result can be found in [4].

1. The main theorem. Recall the following definitions from [2]: A model Y has property MO.1 if it does not have cosimplicial subsets with only one point in each dimension. It has property MO.2 if it is stable for interior points by codegeneracies. In other words it has property MO.2 if whenever y is an interior point then so is $Y(s)(y)$ for each epimorphism s of Δ (whenever defined). A point y is interior if for each monomorphism w of Δ, if y belongs to $\mathrm{Im}(Y(w))$ then w is an identity. In [2] we proved that MO.1 is equivalent to "every $y \in Y^n$ admits a unique decomposition $y = Y(d)(y')$, where d is a monomorphism of Δ and y' is an interior point". A cosimplicial set with this property was called an E-Z type cosimplicial set.

Given a simplicial set X and a cosimplicial set Y we will say that a pair $(x, y) \in \amalg_n X(n) \times Y(n)$ is *optimal* if x is nondegenerate and y is an interior point.

In the rest of this paper we develop the machinery necessary in order to prove the following theorem.

THEOREM. *Let Y be a cosimplicial set with properties* MO.1 *and* MO.2. *Let X be an arbitrary simplicial set. Then the identification map $\amalg_n X(n) \times Y(n) \to R_Y(X)$, associated with Milnor's relation, establishes a one-to-one correspondence between the set of optimal pairs and the set $R_Y(X)$.*

2. Lemmas. We begin by generalizing Milnor's maps M_1, $M_2\colon \amalg_n X(n) \times Y(n) \to \amalg_n X(n) \times Y(n)$. $M_1(x, y)$ is defined as follows: let $y = Y(d)(y')$ be the Eilenberg-Zilber decomposition (E-Z decomposition) of y (i.e. $d =$ mono, $y' =$ interior). This decomposition is unique by property MO.1 of the model Y. One takes $M_1(x, y) = (X(d)(x), y')$. For M_2 let $x = X(s)(x')$ be the E-Z decomposition of X (i.e. $S =$ epimorphism of Δ, $x' =$ nondegenerate) which exists and is unique without any conditions over X. Then $M_2(x, y) = (x', Y(s)(y))$. Now Milnor's function M is the composite $M_2 \cdot M_1$. By property MO.2 it is clear that $M(x, y)$ is an optimal pair. Notice also that $M(x, y) = (x, y)$ when (x, y) is an optimal pair.

We will use the notation $(x, y) \xrightarrow{(w)} (x', y')$ when w is an arrow of Δ such that $X(w)(x) = x'$ and $Y(w)(y') = y$. To prove the theorem, it remains to show that the only optimal pair in the class of (x, y) is $M(x, y)$. It is done through the following lemmas.

2.1 LEMMA. (a) *If d is a monomorphism of Δ and in $(\bar{x}, \bar{y}) \xrightarrow{(d)} (a, b)$, $(\bar{x}, \bar{y})$ is optimal, then d is an identity of Δ and therefore $(\bar{x}, \bar{y}) = (a, b)$.*

(b) *If s is an epimorphism of Δ and in $(a, b) \xrightarrow{(s)} (\bar{x}, \bar{y})$, $(\bar{x}, \bar{y})$ is optimal, then s is an identity of Δ and therefore $(a, b) = (\bar{x}, \bar{y})$.*

2.2 LEMMA. *Let $(\bar{x}, \bar{y})$ denote an optimal pair. Then* (a) *if there exists $(a, b) \xrightarrow{(w)} (\bar{x}, \bar{y})$, then w must be a monomorphism.*

(b) *If there exists $(\bar{x}, \bar{y}) \xrightarrow{(w)} (a, b)$ then w must be an epimorphism of Δ.*

PROOF. (a) By the Mac Lane decomposition of $w = ds$ with d a monomorphism and s an epimorphism of Δ, then $(w) = (s) \cdot (d)$ (in the obvious way). s being an identity (by Lemma 2.1) the result follows. The dual argument can be used for part (b).

2.3 LEMMA. *In* (a) *and* (b) *of Lemma* 2.2 $M(a, b) = (\bar{x}, \bar{y})$.

PROOF. (a) Since w is a monomorphism and $\bar{y}$ is an interior point the E-Z decomposition of b is $Y(w)(\bar{y})$, and $M_1(a, b) = (X(w)(a), \bar{y})$. Since $X(w)(a) = \bar{x}$, $M_1(a, b) = (\bar{x}, \bar{y})$ and then $M(a, b) = (\bar{x}, \bar{y})$. (b) The proof is *not* dual to (a). Since w is an epimorphism (Lemma 2.2(b)), if $b = Y(d)(y')$ is the E-Z decomposition of b then $\bar{y} = Y(wd)(y')$. If $wd = d's'$ is the Mac Lane decomposition of wd then in $\bar{y} = Y(d')(Y(s')(y'))$ d' must be an identity since $\bar{y}$ is an interior point. Thus $wd = s'$ and $\bar{y} = Y(s')(y')$. We apply M to the pair (a, b):

$$(a, b) = (a, Y(d)(y')) \xrightarrow{M_1} (X(d)(a), y') = (X(d)X(w)(\bar{x}), y')$$

$$= (X(s')(\bar{x}), y') \xrightarrow{M_2} (\bar{x}, Y(s')(y')) = (\bar{x}, \bar{y}).$$

Notice that $X(s')(\bar{x})$ is already in its E-Z form.

2.4 LEMMA. *If in the diagram* $(\bar{x}, \bar{y}) \xleftarrow{(d)} (a, b) \xrightarrow{(w)} (c, h)$ *the pair* $(\bar{x}, \bar{y})$ *is optimal, then* $M(c, h) = (\bar{x}, \bar{y})$.

PROOF. From the diagram, d is a monomorphism and $b = Y(d)(\bar{y}) = Y(w)(h)$. Let $h = Y(t)(y)$ be the E-Z decomposition of h, and $d's'$ be the Mac Lane decomposition of wt. Then $b = Y(wt)(y) = Y(d')(Y(s')(y))$ is the E-Z decomposition of b because d' is a monomorphism and $Y(s')(y)$ is interior by MO.2. Since $b = Y(d)(\bar{y})$ is also a E-Z decomposition of b it follows that $d = d'$ and $\bar{y} = Y(s')(y)$. Now

$$(c, h) = (c, Y(t)(y)) \xrightarrow{M_1} (X(t)(c), y) = (X(wt)(a), y) = (X(ds')(a), y)$$

$$= (X(s')(\bar{x}), y) \xrightarrow{M_2} (\bar{x}, Y(s')(y)) = (\bar{x}, \bar{y}).$$

2.5 LEMMA. *If in* $(c, h) \xrightarrow{(d)} (a, b)$ *d is a monomorphism then* $M_1(c, h) = M_1(a, b)$ *and therefore* $M(c, h) = M(a, b)$.

PROOF. Let $b = Y(t)(y)$ be the E-Z decomposition of b. Then $h = Y(dt)(y)$ which is the E-Z decomposition of h. Thus,

$$(c, h) = (c, Y(dt)(y)) \xrightarrow{M_1} (X(dt)(c), y) = (X(t)(a), y) = M_1(a, b).$$

2.6 LEMMA. *If in the diagam* $(\bar{x}, \bar{y}) \xrightarrow{(s)} (a, b) \xleftarrow{(u)} (c, h)$, $(\bar{x}, \bar{y})$ *is an optimal pair and u is an epimorphism then there exists an epimorphism s' in Δ which induces an arrow* $(\bar{x}, \bar{y}) \xrightarrow{(s')} (c, h)$ *and consequently* $M(c, h) = M(a, b) = (\bar{x}, \bar{y})$.

PROOF. Let s' be the epimorphism of the E-Z decomposition of c: $c = X(s')(c')$. Notice that $a = X(s)(\bar{x})$ and $a = X(s'u)(c')$ are E-Z decompositions of a, therefore $s = s'u$ and $c' = \bar{x}$. Now since $X(s')(\bar{x}) = X(s')(c') = c$ and $\bar{y} = Y(s)(b) = Y(s'u)(b) = Y(s')(h)$ then $(\bar{x}, \bar{y}) \xrightarrow{(s')} (c, h)$ is an arrow.

2.7 LEMMA. *If in a diagram* $(\bar{x}, \bar{y}) \xrightarrow{(s)} (a, b) \xleftarrow{(w)} (c, h)$, $(\bar{x}, \bar{y})$ *is an optimal pair, then there exists a diagram* $(\bar{x}, \bar{y}) \xrightarrow{(s')} (a', b') \xleftarrow{(d)} (c, h)$ *where* s' *is an epimorphism and* d *a monomorphism. Thus* $M(c, h) = (\bar{x}, \bar{y})$.

PROOF. Let $w = dv$ be the Mac Lane decomposition of w. One gets a diagram

$$\begin{tikzcd}
 & (a, b) & & \\
(\bar{x}, \bar{y}) \arrow[ur, "(s)"] \arrow[rr, "(s')"'] & & (a', b') \arrow[ul, "(v)"'] & (c, h) \arrow[l, "(d)"']
\end{tikzcd}$$

Note that s' exists by Lemma 2.6.

Summarizing, if in diagrams of the kind $(x, y) \to (x', y') \leftarrow (x'', y'')$ and $(x, y) \leftarrow (x', y') \to (x'', y'')$ one of the extremes is optimal then it is the value by M of the other extreme. So, in order to prove the theorem it is sufficient to prove that if (x, y) is equivalent to (x', y') then $M(x, y) = M(x', y')$, which reduces to proving the statement when there exists an arrow $(x, y) \xrightarrow{(w)} (x', y')$. We consider the diagram $(x', y') \xrightarrow{(d)} M_1(x', y') \xleftarrow{(s)} M(x', y')$ of the definition of M_1 and M_2. Therefore in the diagram $(x, y) \xrightarrow{(dw)} M_1(x', y') \xleftarrow{(s)} M(x', y')$ one of the extremes is optimal and thus $M(x, y) = M(x', y')$.

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