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CONSTRUCTION OF SPACES WITH A σ-MINIMAL BASE

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ABSTRACT. Using product spaces, a method is given for constructing and recognizing certain spaces with a σ-minimal base. This technique shows that every topological space can be embedded as a closed subspace of a space with a σ-minimal base; hence a σ-minimal base, by itself, does not imply any nontrivial closed hereditary topological property. It is also shown that any space Y can be expressed as the open perfect image of some space with a σ-minimal base. Examples are given, illustrating a surprisingly large class of product spaces with a σ-minimal base.

A base $\mathfrak{B}$ for a topological space X is said to be a *σ-minimal base* if $\mathfrak{B} = \bigcup_{n<\omega} \mathfrak{B}_n$ where each $\mathfrak{B}_n$ is a minimal cover of $\bigcup\{B: B \in \mathfrak{B}_n\}$. This concept was introduced by C. Aull in [$\mathbf{A}_1$] where it was noted that every quasi-developable space has a σ-minimal base, and it was asked in [$\mathbf{A}_2$] whether the converse was true. Bennett and Berney answered this question negatively by showing (among other examples) [**BB**] that the lexicographic square had a σ-minimal base. Furthermore they show that the "top and bottom" subspace does not have a σ-minimal base so the σ-minimal base property is not closed hereditary. In this note we give a technique for constructing (and recognizing) spaces with a σ-minimal base and use this to show that any topological space can be embedded as a closed subspace of a space with a σ-minimal base. The question of preservation under perfect mappings is also answered negatively by showing that any space can be expressed as the perfect image of a space with a σ-minimal base.

All regular spaces are assumed to be T_1 but no other separation axioms are assumed unless otherwise stated. The cardinality of a set A is denoted by $|A|$, and the *weight* of a space X, denoted by $w(X)$, is $\min\{|\mathfrak{B}| \cdot \omega: \mathfrak{B}$ is a base for $X\}$. An ordinal number is the set of smaller ordinals and when used as a topological space, the order topology is assumed.

The most useful form of our construction method is given by Theorem 2; the first lemma is used mainly to assist in the proof of Theorem 2.

LEMMA 1. *Suppose X satisfies*:

(a) *$X = A \cup B$ where A and B are open in X*;

(b) *there exist relatively discrete subsets H and K of X such that $H \subset X - A$, $K \subset X - B$, and $|H| = |K| = w(X)$.*

If Y is any space with $w(Y) \leqslant w(X)$ then $Z = Y \times (\prod_{n<\omega} X)$ has a σ-minimal base.

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PROOF. Assume $w(X) = m$, for an infinite cardinal number m, and express $H = \{h_\alpha: \alpha < m\}$ and $K = \{k_\alpha: \alpha < m\}$ where $h_\alpha \neq h_\beta$ and $k_\alpha \neq k_\beta$ if $\alpha \neq \beta$. For each $\alpha < m$, let U_α and V_α be open sets in X such that $U_\alpha \cap H = \{h_\alpha\}$, $V_\alpha \cap K = \{k_\alpha\}$, $A \subset U_\alpha$, and $B \subset V_\alpha$. Note that $\{U_\alpha: \alpha < m\}$ and $\{V_\alpha: \alpha < m\}$ are minimal collections, and $X = U_\alpha \cup V_\alpha$ for each $\alpha < m$. For $i = 1, 2$ and $n < \omega$ we define a minimal collection $\mathcal{G}_{ni}$ as follows.

For $0 < n < \omega$ the space $Y \times (\prod_{j=1}^{n} X)$ has a base of cardinality m which can be expressed as: $\mathcal{W}_n = \{W_{0\alpha} \times W_{1\alpha} \times \cdots \times W_{n\alpha}: \alpha < m\}$ for some choices $W_{0\alpha}, \ldots, W_{n\alpha}$ in the appropriate spaces. Define

$$\mathcal{G}_{n1} = \{W_{0\alpha} \times \cdots \times W_{n\alpha} \times U_\alpha \times X \times X \times \cdots : \alpha < m\},$$
$$\mathcal{G}_{n2} = \{W_{0\alpha} \times \cdots \times W_{n\alpha} \times V_\alpha \times X \times X \times \cdots : \alpha < m\}.$$

Then $\mathcal{G}_{n1}$ and $\mathcal{G}_{n2}$ are minimal collections of open sets in Z and $\mathcal{G} = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{2} \mathcal{G}_{ni}$ is a σ-minimal collection. To see that $\mathcal{G}$ is a base for Z, let $z \in T \subset Z$ where T is open in Z. There is some positive n and $\beta < m$ such that $z \in W_{0\beta} \times \cdots \times W_{n\beta} \times X \times X \times \cdots \subset T$. It is clear that either $z \in W_{0\beta} \times \cdots \times W_{n\beta} \times U_\beta \times X \times X \times \cdots \subset T$ or $z \in W_{0\beta} \times \cdots \times W_{n\beta} \times V_\beta \times X \times X \times \cdots \subset T$. Hence the lemma is proved.

THEOREM 2. *If X is a regular space and X contains a relatively discrete subspace E with $|E| = w(X) \geqslant w(Y)$ then $Z = Y \times (\prod_{n<\omega} X)$ has a σ-minimal base.*

PROOF. To prove this it suffices to show that $X \times X$ satisfies the conditions given for X in Lemma 1. Let $x_0 \in E$ and let U and V be open sets in X such that $x_0 \in U \subset \overline{U} \subset V$ and $V \cap E = \{x_0\}$. Pick $x_1 \in E - V$. Let $A = V \times X$, $B = (X - \overline{U}) \times X$, $H = \{x_1\} \times E$, and $K = \{x_0\} \times E$. Then A, B, H, K satisfy the desired conditions.

COROLLARY 3. *Any space Y can be embedded as a closed subspace of a space Z with a σ-minimal base.*

PROOF. If Y is given let X be a discrete space with $|X| = w(Y)$; then $Y \times (\prod_{n<\omega} X)$ has a σ-minimal base and contains a copy of Y as a closed subspace.

Notice that the space Z, in the above corollary, can often be chosen to possess some of the same topological properties that Y has. For example, if Y is first countable then Z can be chosen to be first countable, and if Y is compact then Z can be chosen to be compact. On the other hand, Corollary 3 also says that, without additional assumptions, a space with a σ-minimal base cannot be expected to naturally possess any (nontrivial) closed hereditary topological property. This should be compared with the situation in GO (generalized ordered) spaces, where Bennett and Lutzer have shown **[BL]** that any GO space with a σ-minimal base is hereditarily paracompact. This provides an easy way to see that ω_1 does not have a σ-minimal base.

Lemma 1 and Theorem 2 were given in their present form for simplicity and ease of proof. It should be remarked that each of these results are true in more general situations involving different spaces X_n, $n < \omega$, instead of copies of a fixed space X. For example, Theorem 2 can be strengthened as follows, using a similar proof.

THEOREM 4. *If X_n is a regular space for $n < \omega$, and X_n contains a relatively discrete subspace E_n with $|E_{n+1}| \geqslant |E_n| = w(X_n) \geqslant w(Y)$ then $Y \times (\prod_{n<\omega} X_n)$ has a σ-minimal base.*

Although base axioms are quite often preserved under perfect mappings we see by the next corollary that this is far from true for spaces with a σ-minimal base. Recall that a continuous onto mapping $f\colon Z \to Y$ is said to be *perfect* if f is closed and $f^{-1}(y)$ is compact in Z for every $y \in Y$.

COROLLARY 5. *A σ-minimal base is not necessarily preserved under a perfect mapping–in fact, if Y is any space then Y is the image, under an open perfect mapping, of some space with a σ-minimal base.*

PROOF. Given the space Y, let X be the one-point compactification of a discrete space of cardinality $w(Y)$. Then $Y \times (\prod_{n<\omega} X)$ has a σ-minimal base and the open projection mapping $p\colon Y \times (\prod_{n<\omega} X) \to Y$ is a perfect mapping since $\prod_{n<\omega} X$ is compact.

Aside from using Theorem 2 for the original purpose of constructing spaces with a σ-minimal base this result can also be used to recognize the presence of a σ-minimal base in many existing product spaces. In the examples below the space Y of Theorem 2 may be thought of as a one-point space.

EXAMPLE 6. The product space $\prod_{n<\omega} \omega_1$ has a σ-minimal base even though ω_1 does not.

EXAMPLE 7. If S denoes the Sorgenfrey line then $\prod_{n<\omega} S$ has a σ-minimal base (and S does not).

PROOF. Apply Theorem 2 using $X = S \times S$. S cannot have a σ-minimal base since S is hereditarily Lindelöf and such a space with a σ-minimal base would have to be second countable [$\mathbf{A}_1$]. It is interesting to note that $S \times S$ has a σ-minimal base, but this is more difficult to verify than the countable product case and will be left to the interested reader.

EXAMPLE 8. The space R^R, with the product topology, has a σ-minimal base.

PROOF. R^R is homeomorphic to $\prod_{n<\omega} X$ where $X = R^R$. As such, X contains a discrete subspace E (use $E = \{f \in R^R\colon f(\alpha) = 1$ for exactly one $\alpha \in R$ and $f(\beta) = 0$ otherwise$\}$), where $|E| = c$, and $w(X) = c$. Apply Theorem 2 and the proof is complete.

Example 8 illustrates a general situation involving "large products". A similar argument verifies our last result.

THEOREM 9. *If $\{Z_\alpha\colon \alpha \in \Lambda\}$ is any collection of regular spaces (at least two points) where $|\Lambda| \geqslant w(Z_\alpha)$ for each $\alpha \in \Lambda$ then $\prod_{\alpha\in\Lambda} Z_\alpha$ has a σ-minimal base.*

PROOF. Let $\{\Lambda_1, \Lambda_2, \cdots\}$ be a countably infinitely partition of Λ where $|\Lambda| = |\Lambda_n|$ for each n. Let $X_n = \prod_{\alpha\in\Lambda_n} Z_\alpha$. Now each X_n contains a discrete subspace E_n with $|E_n| = |\Lambda| = w(X_n)$. Theorem 4 now completes the proof.

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