PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 2, February 1981

A REMARK ON THE MULTIPLICITY OF THE DISCRETE SPECTRUM OF CONGRUENCE GROUPS

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ABSTRACT. It is shown that the discrete spectrum of principal congruence groups of high level can contain eigenvalues of large multiplicity.

Let H^+ = the Poincaré upper half-plane, and let Γ_n = the principal congruence group of level n, considered as a group of isometries of H^+. It is the purpose of this note to point out a very simple observation which nevertheless seems not to be well-known. Namely, there can exist eigenvalues with rather large multiplicities in the discrete spectrum of the Laplacian on $\Gamma_n \setminus H^+$. We illustrate things in the case $n = p$ a prime $\geqslant 3$. For other values of n, one can utilize results of, e.g., Praetorius [2].

THEOREM. *With notation as above, there exist infinitely many* λ *in the discrete spectrum of* Γ_p *having multiplicity at least* $\frac{1}{2}(p + (-1)^{(p-1)/2})$.

REMARK. Hejhal has learned that Selberg has obtained a similar result. Additionally, there seem to be interesting points of contact with a recent paper of Venkov [3].

PROOF OF THE THEOREM. Γ_p is normal in Γ_1, with quotient the simple group $S = \mathrm{PSL}(2, Z_p)$, which is in an obvious way a subgroup of the group of isometries of $\Gamma_p \setminus H^+$. Suppose λ is an eigenvalue in the discrete spectrum of Γ_p, and let E_λ be the corresponding eigenspace. Then S acts unitarily on E_λ and this gives a representation T of S on E_λ. If we decompose T into irreducibles and recall that apart from the trivial representation, the smallest degree which can occur is $\frac{1}{2}(p + (-1)^{(p-1)/2})$ [1], we find that if T contains a single nontrivial irreducible representation, then the dimension of E_λ must be at least $\frac{1}{2}(p + (-1)^{(p-1)/2})$. It certainly suffices therefore to show that T is not a multiple of the trivial representation for all E_λ from some point on. If this happened to be the case, then all the eigenfunctions corresponding to the discrete spectrum of Γ_p would from some point on be automorphic for Γ_1, so we must show that this does not happen. To show this, Hejhal has suggested the following argument, which is simpler than my original approach.

It has been shown by Selberg, and independently by Venkov, that the Weyl asymptotic estimate $N(T) \sim (4\pi)^{-1}(\mathrm{vol}(\Gamma_n \setminus H^+))T$ holds for the discrete spectrum of principal congruence groups (cf. Hejhal's forthcoming book, *The Selberg*

Received by the editors April 11, 1980.
1980 *Mathematics Subject Classification*. Primary 58G25.

trace formula for PSL(2, R), vol. 2, Springer, New York). Since $\mathrm{vol}(\Gamma_p \setminus H^+) = \frac{1}{2}p(p^2 - 1)\mathrm{vol}(\Gamma_1 \setminus H^+)$, this immediately establishes the result.

REMARK. The argument of course shows somewhat more. Namely, that such multiplicities occur with some regularity. A good determination of the intervals within which one can be sure of having such eigenvalues depends on the true order of magnitude of the error term in the Weyl estimate.

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