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THE BRAUER GROUP IS TORSION

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ABSTRACT. We present a new proof that if A is an Azumaya algebra over a commutative ring R of rank n^2, then $A^n = A \otimes_R \cdots \otimes_R A$ is a split Azumaya algebra $\mathrm{End}_R(P)$. We provide a description of P, including that it is a direct summand of A^n.

In [3], a well-known fact about the Brauer group of a field was generalized to show that the Brauer group of a commutative ring was torsion. In fact, if A is Azumaya of constant rank n^2 over R, then A^n $(= A \otimes_R \cdots \otimes_R A) \cong \mathrm{End}_R(P)$ for P an progenerator. In this note we will present a relatively elementary proof of this theorem, and we will also describe P as a specific direct summand of A^n. It should be noted that our argument is related to one that has been used in the case R is a field (see, for example, [6]).

Our proof begins much like the standard one, with faithfully flat splittings. Fix A and R as above. There is a faithfully flat commutative ring extension $R \subset S$ such that $A \otimes_R S \cong M_n(S) \cong \mathrm{End}_S(V)$ where V is a free S module of rank n (e.g. [4, p. 106]). We call such an S and V a free splitting of A. Given such a free splitting, we will identify $A \otimes_R S$, $M_n(S)$ and $\mathrm{End}_S(V)$. Also, if $r > 1$ is an integer, then $A^r \otimes_R S$ is naturally isomorphic to $\mathrm{End}_S(V^r) = \mathrm{End}_S(V \otimes_S \cdots \otimes_S V)$. Identify $A^r \otimes_R S$ with this endomorphism ring.

Let tr: $A \to R$ be the reduced trace map. In [4, p. 112] (result attributed to Goldman), the reduced trace map is used to define a very useful element $\alpha \in A \otimes_R A$. In fact, α is uniquely defined by the property that $\alpha = \sum x_i \otimes y_i$ where $\mathrm{tr}(a) = \sum x_i a y_i$ for all $a \in A$. The properties of α are listed in the next lemma.

LEMMA 1. (a) $\alpha^2 = 1$.

(b) *$\alpha(a \otimes b) = (b \otimes a)\alpha$ for all $a, b \in A$.*

(c) *Let S, V be any free splitting of A. Consider*

$$\alpha \otimes 1 \in A^2 \otimes_R S = \mathrm{End}_S(V \otimes V).$$

Then $\alpha \otimes 1$ is the map defined by $\alpha \otimes 1(v \otimes w) = w \otimes v$.

PROOF. Parts (a) and (b) are directly from [4]. As for (c), the uniqueness of α is used to show that $\alpha \otimes 1 = \sum_{i,j} e_{ij} \otimes e_{ji}$, where the e_{ij} are any matrix units for $A \otimes_R S$. Translated into maps, that is just (c).

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The above facts can be generalized to higher tensor powers of A as follows. Let m be an integer $m \leqslant n$. The symmetric group, S_m, acts on $A^m = A \otimes_R \cdots \otimes_R A$ in the natural way. If S and V are a free splitting of A, $A^m \otimes_R S$ has been identified with $\mathrm{End}_S(V^m)$. The natural action of S_m on V^m induces the action of S_m on $A^m \otimes_R S$.

THEOREM 2. *For each $\sigma \in S_m$ there is a unit $\alpha_\sigma \in A^m$ such that*

(a) $\alpha_\sigma^{-1}\beta\alpha_\sigma = \sigma(\beta)$ *for all* $\beta \in A^m$.

(b) *The map $\sigma \to \alpha_\sigma$ is a homomorphism from S_m to the group of units of A^m.*

(c) *If S and V are a free splitting of A, then $\alpha_\sigma \otimes 1 \in A^m \otimes_R S$, considered as an endomorphism of V^m, is just the map σ.*

(d) *The α_σ's are linearly independent over R.*

PROOF. For a fixed S and V, we use (c) to define $\alpha'_\sigma \in A^m \otimes_R S$. To prove parts (a) and (b), it suffices to show that the α'_σ are in the image of A^m. As S_m is generated by 2-cycles, we may assume σ is a 2-cycle. But after obvious identifications, this case is covered by Lemma 1. Part (c) for arbitrary S and V follows from 1(c). As for (d), it suffices to prove that the $\alpha_\sigma \otimes 1$'s are linearly independent over S, and this follows from the easy observation that V^m is a faithful module over the group algebra $S(S_m)$. Q.E.D.

We now turn to the algebra A^n, where we recall that A has rank n^2 over R. In A^n, we define $\beta = \sum_{S_n}\mathrm{sgn}(\sigma)\alpha_\sigma$. Of course, $\mathrm{sgn}(\sigma)$ is ± 1 depending on whether the permutation σ is even or odd. Let S and V be any free splitting of A. Consider $\beta \otimes 1$ as an S endomorphism of V^n. $(\beta \otimes 1)(x_1 \otimes \cdots \otimes x_n) = 0$ if $x_i = x_j$ for $i \neq j$. If $v_1, \ldots, v_n$ are an S basis of V, we quickly see that $(\beta \otimes 1)(V^n)$ is generated over S by the single element $w = \sum_{S_n}\mathrm{sgn}(\sigma)v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$. What is more, w is part of a free basis of V^n. Let W be the kernel of $\beta \otimes 1$, so W is a direct summand of V^n. The left ideal $(A^n \otimes_R S)(\beta \otimes 1)$, as a subset of $\mathrm{End}_S(V^n)$, is exactly those endomorphisms which are zero on W. Now set $J = A^n\beta$. Then $(A^n/J) \otimes_R S \cong (A^n \otimes_R S)/(A^n \otimes_R S)(\beta \otimes 1) \cong \mathrm{Hom}_S(W, V^n)$ as S modules, so $(A^n/J) \otimes_R S$ is an S progenerator. Since S is faithfully flat over R, A^n/J is an R progenerator (e.g. [**1**, p. 34]). Since A^n is Azumaya over R. A^n/J is a projective A^n module and so J is an A^n direct summand. We have proved most of the first part of the following theorem.

THEOREM 3. (a) *$J = A^n\beta$ is a A^n-direct summand of A^n, and an R progenerator of rank n^n.*

(b) $A^n \cong \mathrm{End}_R(J)$.

PROOF. Part (a) has been shown, except for the trivial calculation of the rank of J. As for (b), J is a faithful A^n module because it is a faithful R module (e.g. [**2**, p. 54]). The injection $A^n \to \mathrm{End}_R(J)$ must be surjective using the double centralizer theorem and the equal R ranks of A^n and $\mathrm{End}_R(J)$ (e.g. [**2**, p. 57]). Q.E.D.

As a final remark let us note that the proof of Theorem 3 is a special case of more general phenomenon. If B is an Azumaya algebra over a field F of dimension n^2, the rank of any $b \in B$ can be unambiguously defined as $(1/n)(\dim_F Bb)$. Let A

be an Azumaya algebra over R, with rank n^2, and let $\mathfrak{P} \subseteq R$ be a prime ideal of R. For any $a \in A$, we define the rank of a at $\mathfrak{P}$ to be the rank of $a \otimes 1 \in A \otimes_R K$, where K is the field of quotients of $R/\mathfrak{P}$. We say $a \in A$ has constant rank r if a has rank r at every prime ideal of R. Using arguments as in [5, p. 339], one can show that if a has constant rank r then Aa is an R progenerator of rank nr and an A direct summand of A. The element β used above has constant rank one.

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