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THE STRUCTURE OF ω-LIMIT SETS OF NONEXPANSIVE MAPS

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ABSTRACT. It is shown that ω-limit sets of nonexpansive maps carry the structure of monothetic topological groups. This fact is then used to answer a question of Edelstein.

Let T: $D \to D$ be a map from a subset D of a metric space X into itself. We call T *nonexpansive* if $d(T(x), T(y)) \leqslant d(x, y)$ for all x and y in D. For x in X, we call the set $\gamma(x) = \bigcup_{n \in \mathbf{Z}^+} T^n(x)$ the *orbit* of x, while the (possibly empty) set $\omega(x) = \{y \in X: y = \lim_{N_1} T^n(x),\ N_1$ a strictly increasing sequence in $\mathbf{Z}^+\}$ is called the *ω-limit* set of x. A subset Ω of X is called *minimal* under T if $\Omega = \operatorname{cl} \gamma(y)$ for each y in Ω, and is called *strongly invariant* under T if for each n in $\mathbf{Z}^+$ the map T^n is a homeomorphism of Ω onto itself.

If T is a nonexpansive selfmap of the closed subset C of X, then it is known from the theory of dynamical systems [4] that a nonempty ω-limit set is actually minimal, and that the semigroup $\{T^n: n \in \mathbf{Z}^+\}$ may be extended to a group of homeomorphisms $\{T^n: n \in \mathbf{Z}\}$ on this set. Further algebraic structure is provided by the following.

THEOREM 1. *Let C be a closed set in the Banach space X and let T: $C \to C$ be nonexpansive. If for some $x \in C$ the ω-limit set G of x is nonempty, then there exists a binary operation in G under which it is a monothetic topological group in the topology induced by the metric of X. (Recall a topological group G is called monothetic if it contains an element x such that $\{x^n: n \in \mathbf{Z}\}$ is dense in G.)*

PROOF. If G is nonempty, then as before it is minimal and strongly invariant under T, and T^n is an isometric homeomorphism of G for all n (see [4, Theorem 1]). Choose $e \in G$ arbitrarily and define a binary operation on G (denoted by juxtaposition) as follows: for any $b \in G$, find a subsequence N_b of $\mathbf{Z}^+$ such that $\lim_{N_b} T^n(e) = b$; then for $a \in G$ define $ab = \lim_{N_b} T^n(a)$. Using the fact that T is isometric on G, it is straightforward to show that this limit exists and is unique. In fact, by applying a theorem of Moore [7, p. 100] on double sequences, it can be shown that this operation is abelian. Now the element e is clearly the identity, and since T is a homeomorphism on G, inverses are given by the formula $b^{-1} = \lim_{N_b} T^{-n}(e)$. Associativity follows directly from the definition. Thus G is a group.

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Suppose $a_n \to a$, $b_n \to b$ are sequences with all elements belonging to G. Then

$$\begin{aligned}\|a_n b_n^{-1} - ab^{-1}\| &< \|b_n^{-1}a_n - b^{-1}a_n\| + \|a_n b^{-1} - ab^{-1}\| \\ &= \Big\|\lim_{N_{a_n}} T^n b_n^{-1} - \lim_{N_{a_n}} T^n b^{-1}\Big\| + \Big\|\lim_{N_b} T^{-n} a_n - \lim_{N_b} T^{-n} a\Big\| \\ &= \|b_n^{-1} - b^{-1}\| + \|a_n - a\|\end{aligned}$$

which is small for large enough n. The operation is therefore continuous in both variables and G is a topological group.

REMARKS. Della Riccia [5] has shown that if we assume only an equicontinuous semiflow acting on either a locally compact or a complete metric space, then again a nonempty ω-limit set $\omega(x)$ is minimal and the semiflow may be extended to an equicontinuous flow of homeomorphisms of $\omega(x)$. Our result holds in this setting as well, since in the proof of Theorem 1, equicontinuity of T^n suffices to show existence and uniqueness of the limits. The remainder of the proof carries through directly.

It is known [**8**, p. 84] that if G is a locally compact monothetic group, then G is either compact or topologically equivalent to $\mathbf{Z}$. Unfortunately, this result can fail outside the locally compact setting. Rolewicz [**9**] has described a monothetic complete metric abelian group which is neither discrete nor compact. One may also construct an example in $L_2(\mu)$ where μ is a nonatomic probability measure on the circle. Choose μ so that $\lim \sup|\hat{\mu}| = 1$ and set $Uf = e^{i\theta}f$ in $L_2(\mu)$. Then ω-limit sets of U have the desired properties.

Examples are also available from constructions of Edelstein [**6**]. He has given an example of a fixed point free isometry of Hilbert space which has a point whose ω-limit set is nonempty and unbounded. The argument of the present paper shows this ω-limit set must have been non-locally-compact.

However, we may use Theorem 1 to give an affirmative answer to a question posed by Edelstein [**10**, p. 66].

THEOREM 2. *Let X be a finite-dimensional, not necessarily strictly convex, Banach space, and let D be an arbitrary closed subset of X. If $\omega(x)$ is nonempty for some x in X, then every orbit is bounded.*

PROOF. X is now locally compact, so $G = \omega(x)$ is either compact or isomorphic to $\mathbf{Z}$. But G is recurrent, so it must be compact, hence bounded. By nonexpansivity, each orbit is also bounded.

REMARKS. We note that different arguments obtain this result in the strictly convex case [**6**]. If D is closed in $\mathbf{R}^n$, and is in addition convex, then by standard arguments we easily obtain the existence of fixed points. Thus a fixed point free nonexpansive map on all of $\mathbf{R}^n$ has the property that $T^n(x) \to \infty$ for all x. This formulation of the result is reminiscent of the Brouwer Translation Theorem: if T is a homeomorphism of E^2 onto itself which preserves orientation and has no fixed points, then $T^n(x) \to \infty$ for all x as $n \to \pm\infty$ (see [**1**] or [**3**]).

For D closed but not convex, the obvious example of a rotation of the circle shows that we cannot expect fixed points. If D is assumed topologically trivial, however, one may still hope for a fixed point. We do not know if such a result holds, but can assert that it cannot be valid for topological reasons alone. For Conner and Floyd (see [2, p. 58]) have constructed a periodic transformation of $\mathbf{R}^n$ which is fixed point free. For such a map an equivalent metric can be defined so that the map is a periodic isometry. Clearly this new metric cannot be a Banach space norm on $\mathbf{R}^n$ since a nonempty convex invariant set could easily be produced.

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