ON NEAR-DERIVATIONS

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Abstract. In this note we show how near-derivations can be expressed by biadditive and additive functions satisfying further conditions.

1. The concept of near-derivation has been introduced and discussed by Lawrence, Mess and Zorzitto [3] in connection with nonnegative information functions. A real-valued function $\gamma$ defined on the reals $\mathbb{R}$ is called a near-derivation if

$$\gamma(xy) = xy\gamma(y) + y\gamma(x) \quad \text{for all } x, y \in \mathbb{R}, \quad (1)$$

$$\gamma(x + y) \geq \gamma(x) + \gamma(y) \quad \text{for all } x \geq 0, y \geq 0, \quad (2)$$

$$\gamma(r) = 0 \quad \text{for all rational } r. \quad (3)$$

Daróczy and Maksa showed in [1] that there exists a near-derivation which is not a derivation. Namely, if $d: \mathbb{R} \to \mathbb{R}$ is a nonidentically zero derivation then the function $\gamma$ defined by

$$\gamma(x) = \begin{cases} d(d(x)) - d(x)^2/x & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases} \quad (4)$$

is such a near-derivation. In this note we present a method for recovering near-derivations in terms of biadditive functions on $\mathbb{R}^2$ and additive functions on $\mathbb{R}$ satisfying further conditions.

2. It has been proved in [3] that for any near-derivation $\gamma$ the finite limit

$$\alpha(x) = \lim_{n \to \infty} \gamma(x + n) \quad (5)$$

exists for all $x \in \mathbb{R}$. Furthermore, the function $\alpha: \mathbb{R} \to \mathbb{R}$ defined by (5) has the following properties:

(a) $\gamma(x) < \alpha(x)$ for all $x > 0$,

(b) $\alpha(x + y) = \alpha(x) + \alpha(y)$ for all $x, y \in \mathbb{R}$,

(c) $2x\alpha(x) < \alpha(x^2)$ for all $x \in \mathbb{R}$,

(d) $\alpha(r) = 0$ for all rational $r$.

Using (a) and (1)–(3) we have

$$ta\left(\frac{1}{t}\right) + \frac{1}{t} \alpha(t) = |t|\alpha\left(\frac{1}{|t|}\right) + \frac{1}{|t|} \alpha(|t|)$$

$$> |t|\gamma\left(\frac{1}{|t|}\right) + \frac{1}{|t|} \gamma(|t|) = \gamma(1) = 0$$

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that is,

\[(e) \, \alpha(1/t) + (1/t)\alpha(t) > 0 \text{ for all } t \in \mathbb{R} \setminus \{0\}.\]

Suppose that the function \(\alpha: \mathbb{R} \to \mathbb{R}\) satisfies (b)–(e) and define \(A: \mathbb{R}^2 \to \mathbb{R}\) by

\[A(x,y) = \alpha(xy) - xa(y) - ya(x).\]  \hfill (6)

It is easy to see that \(A\) has the properties:

\[A(x,y) = A(y,x),\]  \hfill (7)

\[A(x + y, z) = A(x, z) + A(y, z),\]  \hfill (8)

\[A(x, x) > 0,\]  \hfill (9)

\[A(xy, z) + zA(x, y) = A(x, yz) + xA(y, z),\]  \hfill (10)

\[A(t, 1/t) < 0,\]  \hfill (11)

for all \(x, y, z \in \mathbb{R}\) and \(t \in \mathbb{R} \setminus \{0\}\). It follows from [2] that a function \(A: \mathbb{R}^2 \to \mathbb{R}\) satisfying (7)–(11) is always of the form (6) where \(\alpha: \mathbb{R} \to \mathbb{R}\) has the properties (b)–(e).

**Theorem 1.** Suppose that the function \(\alpha: \mathbb{R} \to \mathbb{R}\) satisfies (b)–(e) and define \(A: \mathbb{R}^2 \to \mathbb{R}\) by (6). Then the function \(\gamma\) given by

\[\gamma(x) = \begin{cases} \alpha(x) - \sum_{n=1}^{\infty} 2^{n-1}x^{1/2 - 1/2^n}A(x^{1/2^n}, x^{1/2^n}) & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -\gamma(-x) & \text{if } x < 0 \end{cases}\]  \hfill (12)

is a near-derivation.

**Proof.** Let \(\gamma_n(x) = 2^nx^{1-1/2^n}\alpha(x^{1/2^n})\) for \(x > 0\) and \(n = 0, 1, \ldots\). Then

\[\gamma_n(x) - \gamma_{n-1}(x) = -2^{n-1}x^{1-1/2^n - 1}A(x^{1/2^n}, x^{1/2^n}).\]  \hfill (13)

Hence, by (9), we get that the sequence \((\gamma_n(x))\) is decreasing for all fixed \(x > 0\). According to (c) and (11)

\[\gamma_n(x) + x^2\gamma_{n-1}(1/x) > -2^nxA(x^{1/2^n}, x^{-1/2^n}) > 0\]

and therefore

\[\gamma_n(x) > -x^2\gamma_{n-1}(1/x) > -x^2\gamma_0(1/x),\]

which means that the sequence \((\gamma_n(x))\) is bounded below for all fixed \(x > 0\). Thus \((\gamma_n(x))\) is convergent and its limit is in \(\mathbb{R}\). On the other hand (13) and (12) imply that

\[\gamma(x) = \lim_{n \to \infty} \gamma_n(x)\]  \hfill (14)

holds for all \(x > 0\). By the Cauchy-Schwarz inequality for bilinear forms on a rational vector space

\[|A(u,v)| < \sqrt{A(u,u) \cdot A(v,v)}\]  \hfill (15)

for all \(u, v \in \mathbb{R}\). Let \(x > 0, y > 0\) and \(u = x^{1/2}, v = y^{1/2^n} (n = 0, 1, \ldots)\). Using (6), the definition of \((\gamma_n(x))\) and (13), (15) implies that

\[|\gamma_n(xy) - xy\gamma_n(y) - y\gamma_n(x)| < 2\sqrt{xy} \sqrt{\gamma_{n-1}(x) - \gamma_n(x)} \sqrt{\gamma_{n-1}(y) - \gamma_n(y)}.\]
Hence, by (14), we obtain (1) for all \(x > 0, y > 0\). From (12) and (9) \(\gamma(t) < \alpha(t)\) for \(t > 0\). Thus
\[
\gamma\left(\frac{x}{x + y}\right) + \gamma\left(\frac{y}{x + y}\right) < \alpha\left(\frac{x}{x + y}\right) + \alpha\left(\frac{y}{x + y}\right) = \alpha(1) = 0
\]
for all \(x > 0, y > 0\). By (1) this implies (2) for \(x > 0, y > 0\). To prove (3) let \(r\) be a positive rational number. Then \(A(r, r) = 0\) therefore by (15) \(A(r, u) = 0\) for all \(u \in \mathbb{R}\). Substituting \(x = y = \sqrt{r}\), \(z = 1/\sqrt{r}\) in (10) we see that \(A(\sqrt{r}, \sqrt{r}) = rA(\sqrt{r}, 1/\sqrt{r})\). Thus (9) and (11) give that \(A(\sqrt{r}, \sqrt{r}) = 0\). By induction we have \(A(r^{1/2^n}, r^{1/2^n}) = 0\); thus (12) and (d) imply (3) for all positive rational \(r\). Since \(\gamma\) is an odd function the proof is complete.

**Theorem 2.** Let \(\gamma\) be a near-derivation. Then there exist functions \(\alpha: \mathbb{R} \to \mathbb{R}\) and \(A: \mathbb{R}^2 \to \mathbb{R}\) satisfying (b) and (7)–(11), respectively such that (12) holds for all \(x \in \mathbb{R}\).

**Proof.** We have known that there exists a function \(\alpha: \mathbb{R} \to \mathbb{R}\) with the properties (a)–(e). Thus the function \(A: \mathbb{R}^2 \to \mathbb{R}\) given by (6) satisfies (7)–(11). Define the function \(\delta\) on \(\mathbb{R}\) by
\[
\delta(x) = \begin{cases} 
\alpha(x) - \sum_{n=1}^{\infty} 2^{n-1}x^{1-1/2^n-1}A(x^{1/2^n}, x^{1/2^n}) & \text{if } x > 0, \\
0 & \text{if } x = 0, \\
-\delta(-x) & \text{if } x < 0.
\end{cases}
\]
Applying Theorem 1 we get that \(\delta\) is a near-derivation. Using (6), (a) and (1) we have for \(x > 0\)
\[
\delta(x) = \lim_{n \to \infty} 2^n x^{1-1/2^n} \alpha(x^{1/2^n}) > \lim_{n \to \infty} 2^n x^{1-1/2^n} \gamma(x^{1/2^n}) = \gamma(x).
\]
Since \(\delta - \gamma\) satisfies (1) this implies that \(\delta = \gamma\), thus the proof is complete.

We remark that if \(d\) is a nonidentically zero derivation and \(\alpha(x) = d(d(x))\), \(A(x, y) = 2d(x)d(y)\) \((x, y \in \mathbb{R})\) then Theorem 1 gives the example (4).

**References**


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