NEW SUPPORT POINTS OF $S$ AND EXTREME POINTS OF $K S$

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Abstract. Let $S$ be the usual class of univalent analytic functions $f$ on $\{z \mid |z| < 1\}$ normalized by $f(0) = 1$. We prove that the functions

$$f_{x,y}(z) = \frac{z - \frac{1}{2}(x+y)z^2}{(1-yz)^2}, \quad |x| = |y| = 1, x \neq y,$$

which are support points of $C$, the subclass of $S$ of close-to-convex functions, and extreme points of $D(C_2)$ are support points of $S$ and extreme points of $D(S)$ whenever $0 < |\arg(-x/y)| < \pi/4$. We observe that the known bound of $\pi/4$ for the acute angle between the omitted arc of a support point of $S$ and the radius vector is achieved by the functions $f_{x,y}$ with $|\arg(-x/y)| = \pi/4$.

Introduction. Let $\mathcal{E}$ be the set of analytic functions on the open unit disk. With the usual topology of uniform convergence on compacta $\mathcal{E}$ is a locally convex linear topological space. Suppose $B \subset \mathcal{E}$. A function $b$ in $B$ is called a support point of $B$ if $b$ maximizes $\text{Re} \, J$ over $B$ for some continuous linear functional $J$ on $B$ such that $\text{Re} \, J$ is not constant on $B$. Let $K B$ denote the closed convex hull of $B$. A function $b$ in $K B$ is called an extreme point of $K B$ if $b = t b_1 + (1 - t) b_2$ implies $b = b_1 = b_2$ whenever $0 < t < 1$ and $b_1, b_2 \in K B$.

Let $S$ be the usual class of univalent functions $f$ in $\mathcal{E}$ normalized by $f(0) = 1 + a_2 z^2 + \cdots$. A. Pfluger [10] and L. Brickman and D. R. Wilken [3] have shown that if $f$ is a support point of $S$, then $f$ maps the open unit disk to the complement of an analytic arc $\Gamma$, which tends to $\infty$ with increasing modulus. Furthermore, $\Gamma$ satisfies the $\pi/4$-property, i.e., if $\Gamma$ is oriented so that $\Gamma$ is (positively) traversed from the finite tip to $\infty$, then the angle between the oriented tangent vector to $\Gamma$ and the radius vector to $\Gamma$ at any point is less than or equal to $\pi/4$, with strict inequality at each point of $\Gamma$ except possibly at the finite tip.

In an early paper [1] L. Brickman proved that if $f$ is an extreme point of $K S$, then $f$ maps the open unit disk to the complement of an arc which tends to $\infty$ with increasing modulus. Later W. E. Kirwan and R. W. Pell [9] improved Brickman's result. A special case of their result states that if $f$ is an extreme point of $K S$ and if the omitted arc of $f$ is smooth, then the omitted arc of $f$ satisfies the $\pi/4$-property, albeit, not necessarily with strict inequality.

Since $S$ and $K S$ are compact a lemma in Dunford and Schwartz [5, p. 440] implies that if $f$ is an extreme point of $K S$, then $f \in S$. The following lemma shows that in certain cases we can identify support points of $S$ as extreme points of $K S$.

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Lemma. Let \( J \) be a continuous linear functional on \( \mathcal{S} \) such that \( \text{Re} J \) is nonconstant on \( \mathcal{S} \). If there exist at most two support points of \( \mathcal{S} \) which maximize \( \text{Re} J \) over \( \mathcal{S} \), then each such support point of \( \mathcal{S} \) is an extreme point of \( \mathcal{K}\mathcal{S} \).

It is well known that the Koebe functions \( k_x(z) = \frac{z}{(1 - xz)^2}, |x| = 1 \), uniquely maximize \( \text{Re} J_x \) over \( \mathcal{S} \), where \( J_x g = \bar{g}''(0), |x| = 1 \). Thus, the Koebe functions \( k_x, |x| = 1 \), are both support points of \( \mathcal{S} \) and extreme points of \( \mathcal{K}\mathcal{S} \). Until recently, no other support points of \( \mathcal{S} \) or extreme points of \( \mathcal{K}\mathcal{S} \) were explicitly known. However, J. Brown [4] has determined the support points of \( \mathcal{S} \) which maximize \( \text{Re} J \) over \( \mathcal{S} \), where \( J_g = g(z_0), 0 < |z_0| < 1 \), and that each such support point of \( \mathcal{S} \) is an extreme point of \( \mathcal{K}\mathcal{S} \).

The class \( \mathcal{C} \). Let \( \mathcal{C} \) be the subclass of \( \mathcal{S} \) of close-to-convex functions. In [2] L. Brickman, T. H. MacGregor, and D. R. Wilken showed that the extreme points of \( \mathcal{K}\mathcal{C} \) are the functions

\[
\begin{align*}
fx^x(z) &= \frac{z - \frac{1}{2}(x + y)z^2}{(1 - yz)^2}, \quad |x| = |y| = 1, x \neq y.
\end{align*}
\]

Later E. Grassman, W. Hengartner, and G. Schober [7] proved that each support point of \( \mathcal{C} \) is a function of the form (1). In [8] D. R. Wilken and R. Hornblower showed that each extreme point of \( \mathcal{K}\mathcal{C} \) is a support point of \( \mathcal{C} \).

A natural question arises as to whether the functions (1) are support points of \( \mathcal{S} \) or extreme points of \( \mathcal{K}\mathcal{S} \). Each function \( fx^x \) in (1) maps the open unit disk to the complement of a half-line. Let \( \Gamma_{x^x} \), the omitted half-line of \( fx^x \), be oriented so that \( \Gamma_{x^x} \) is traversed from \( P_{x^x} \), the finite tip of \( \Gamma_{x^x} \), to \( \infty \). A computation shows that \( |\arg(-x/y)| \) is the angle between the tangent vector to \( \Gamma_{x^x} \) and the radius vector to \( \Gamma_{x^x} \) at \( P_{x^x} \). It is easily seen that the angle between the tangent vector to \( \Gamma_{x^x} \) and the radius vector to \( \Gamma_{x^x} \) decreases (monotonically) to 0 as \( \Gamma_{x^x} \) is traversed (monotonically) from \( P_{x^x} \) to \( \infty \). Thus, if \( \pi/4 < |\arg(-x/y)| < \pi \), then \( fx^x \) can be neither a support point of \( \mathcal{S} \) nor an extreme point of \( \mathcal{K}\mathcal{S} \) (because \( \Gamma_{x^x} \) fails to satisfy the \( \pi/4 \)-property). If \( |\arg(-x/y)| = 0 \), i.e., if \( -x = y \), then \( fx^x \) is the Koebe function \( k_y \) and is both a support point of \( \mathcal{S} \) and an extreme point of \( \mathcal{K}\mathcal{S} \). In the remaining case \( 0 < |\arg(-x/y)| < \pi/4 \), \( fx^x \) does not violate the \( \pi/4 \)-property. We will show for \( 0 < |\arg(-x/y)| < \pi/4 \) that \( fx^x \) is both a support point of \( \mathcal{S} \) and an extreme point of \( \mathcal{K}\mathcal{S} \).

To prove the main result of this paper, we recall the bound on \( |\arg f'(z_0)| \) for \( f \) in \( \mathcal{S} \) given by G. M. Goluzin [6, p. 115]. Namely, Goluzin showed that if \( f \in \mathcal{S} \), then

\[
|\arg f'(z_0)| < 4 \arcsin |z_0|, \quad |z_0| < \frac{1}{\sqrt{2}}.
\]

We now prove

Theorem. Let \( fx^x \) be given by (1). If \( 0 < |\arg(-x/y)| < \pi/4 \), then \( fx^x \) is both a support point of \( \mathcal{S} \) and an extreme point of \( \mathcal{K}\mathcal{S} \).
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PROOF. If we differentiate $f_{x,y}$ and then evaluate at $z_0$, we have

$$f'_{x,y}(z_0) = \frac{1 - xz_0}{(1 - yz_0)^3}.$$  

An easy argument shows for $0 < |z_0| < 1$ that

$$|\arg f'_{x,y}(z_0)| < 4 \arcsin|z_0|$$

and that equality occurs in (3) if and only if

$$\arg xz_0 = -\arccos|z_0|, \quad \arg yz_0 = \arccos|z_0|$$  

or

$$\arg xz_0 = \arccos|z_0|, \quad \arg yz_0 = -\arccos|z_0|.$$  

If (4) holds, then $\arg f'_{x,y}(z_0) = 4 \arcsin|z_0|$ and if (5) holds, then $\arg f'_{x,y}(z_0) = -4 \arcsin|z_0|$. We note that for each pair $(x,y)$, $|x| = |y| = 1$, $x^2 \neq y^2$, there exists a unique $z_0$, $0 < |z_0| < 1$, such that exactly one of (4) or (5) holds.

Let $0 < |\arg(-x/y)| < \pi/4$ and suppose $z_0$ satisfies (4). Then (4) implies $0 < |z_0| < \sin \pi/4$. Goluzin’s bound (2) on $|\arg f'(z_0)|$ implies that the region of variability of $f'(z_0)$ for $f$ in $S$ lies in a closed sector in the closed right half-plane. Together (2)–(4) imply that $f'_{x,y}(z_0)$ lies on the upper edge of the region of variability of $f'(z_0)$ for $f$ in $S$. By rotating the region of variability of $f'(z_0)$ for $f$ in $S$ we can realize a continuous linear functional $J_{x,y}$ whose real part is maximized over $S$ by $f_{x,y}$; namely

$$J_{x,y}g = -e^{i(\pi/2 - 4 \arcsin|z_0|)}g(z_0).$$

Similarly, if $0 < |\arg(-x/y)| < \pi/4$ and $z_0$ satisfies (5), then $f_{x,y}$ maximizes $\Re J_{x,y}$ over $S$ where

$$J_{x,y}g = -e^{-i(\pi/2 - 4 \arcsin|z_0|)}g(z_0).$$

We will show now that if $0 < |\arg(-x/y)| < \pi/4$, then $\Re J_{x,y}$ is uniquely maximized over $S$ by $f_{x,y}$, and if $|\arg(-x/y)| = \pi/4$, then $\Re J_{x,y}$ is maximized over $S$ (only) by $f_{x,y}$ and $f_{y,x}$. The lemma will then imply if $0 < |\arg(-x/y)| < \pi/4$, then $f_{x,y}$ is an extreme point of $\mathcal{K}S$.

As in the first part, we can see that if $0 < |z_0| < \sin \pi/8$ and $f^*$ in $S$ maximizes (minimizes) $\arg f'(z_0)$ over $S$, then $f^*$ is a support point of $S$ and, hence, in particular, a slit mapping. Goluzin’s argument [6, p. 115], which shows that (2) is sharp, also shows that for $0 < |z_0| < 1/\sqrt{2}$ there exists a unique slit mapping which maximizes (minimizes) $\arg f'(z_0)$ over $S$.

Let $0 < |\arg(-x/y)| < \pi/4$ and let $z_0$ satisfy (4). Since determining the functions which maximize $\Re J_{x,y}$ over $S$ is equivalent to determining the functions which maximize $\arg f'(z_0)$ over $S$, we conclude from the above that $\Re J_{x,y}$ is uniquely maximized over $S$ by $f_{x,y}$. Similarly, if $0 < |\arg(-x/y)| < \pi/4$ and $z_0$ satisfies (5), then $\Re J_{x,y}$ is uniquely maximized over $S$ by $f_{x,y}$.

Let $|\arg(-x/y)| = \pi/4$ and let $z_0$ satisfy (4) or (5). It is easily seen, from (2)–(5) that one of $f_{x,y}$ and $f_{y,x}$ maximizes $\arg f'(z_0)$ over $S$ and the other minimizes $\arg f'(z_0)$ over $S$. Since, in this case, we have $|z_0| = \sin \pi/8$, it follows that
Thus, determining the functions which maximize \( \text{Re} \, J_{x,y} \) over \( \mathcal{S} \) is equivalent to determining the functions which maximize or minimize \( \arg f'(z_0) \) over \( \mathcal{S} \). Consequently, \( \text{Re} \, J_{x,y} \) is maximized over \( \mathcal{S} \) (only) by \( f_{x,y} \) and \( f_{y,x} \).

**Remark.** For the functions \( f_{x,y} \) with \( |\arg(-x/y)| = \pi/4 \), the known bound of \( \pi/4 \) for the acute angle between the omitted arc of a support point of \( \mathcal{S} \) and the radius vector is achieved (at the finite tip).

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