HOLOMORPHIC MAPS THAT EXTEND TO AUTOMORPHISMS OF A BALL

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Abstract. It is proved, under hypotheses that may be close to minimal, that certain types of biholomorphic maps of subregions of the unit ball in \( \mathbb{C}^n \) have the extension property to which the title alludes.

Let \( B \) (or \( B_n \), when necessary) denote the open unit ball of \( \mathbb{C}^n \). Thus \( z = (z_1, \ldots, z_n) \in B \) provided that \( |z| < 1 \), where \( |z| = \langle z, z \rangle^{1/2} \) and \( \langle z, w \rangle = \sum z_j \bar{w}_j \). An automorphism of \( B \), i.e., a member of \( \text{Aut}(B) \), is, by definition, a holomorphic map of \( B \) onto \( B \) that is one-to-one, and whose inverse is therefore also holomorphic. The sphere that bounds \( B \) is denoted by \( S \).

The following extension theorem will be proved.

**Theorem.** Assume that \( n > 1 \), and that
(a) \( \Omega_1 \) and \( \Omega_2 \) are connected open subsets of \( B \),
(b) for \( j = 1, 2 \), \( \Gamma_j \) is an open subset of \( S \) such that \( \Gamma_j \subset \partial \Omega_j \),
(c) \( F \) is a holomorphic one-to-one map of \( \Omega_1 \) onto \( \Omega_2 \), and
(d) there is a point \( \alpha \in \Gamma_1 \), not a limit point of \( B \cap \partial \Omega_1 \), and a sequence \( \{a_j\} \) in \( \Omega_1 \), converging to \( \alpha \), such that \( \{F(a_j)\} \) converges to a point \( \beta \in \Gamma_2 \), not a limit point of \( B \cap \partial \Omega_2 \).

Then there exists \( \Phi \in \text{Aut}(B) \) such that \( \Phi(z) = F(z) \) for all \( z \in \Omega_1 \).

The relation of this theorem to earlier results will be discussed after its proof.

The proof will use the following well-known facts.

(I) If \( F : B_k \to B_n \) is holomorphic, and \( F(0) = 0 \), then \( |F(z)| < |z| \) for all \( z \in B_k \), and the linear operator \( F'(0) \) (the Fréchet derivative of \( F \) at 0) maps \( B_k \) into \( B_n \).

(II) If, in addition, \( k = n \), then the Jacobian \( JF \) of \( F \) satisfies \( |(JF)(0)| < 1 \); equality holds only when \( F \) is a unitary operator on \( \mathbb{C}^n \).

(III) If \( F \in \text{Aut}(B) \) and \( F(0) = 0 \), then \( F \) is unitary.

Here is a brief indication of how these are proved. For unit vectors \( u \) and \( v \) in \( \mathbb{C}^k \) and \( \mathbb{C}^n \), respectively, the classical Schwarz lemma applies to the function \( g \) defined by

\[
g(\lambda) = \langle F(\lambda u), v \rangle, \quad (\lambda \in \mathbb{C}, |\lambda| < 1).
\]
Thus \(|g(\lambda)| < |\lambda|\) for all eligible \(u, v\), which leads to \(|F(z)| < |z|\), and \(|g'(0)| < 1\), which completes (I), since

\[
g'(0) = \langle F'(0)u, v \rangle.
\]  

(2)

Since (I) implies that no eigenvalue of \(F'(0)\) exceeds 1 in absolute value, it follows that

\[
|(JF)(0)| = |\det F'(0)| < 1.
\]  

(3)

If \(|(JF)(0)| = 1\), then the linear operator \(F'(0)\) preserves volume, and maps \(B\) into \(B\), hence is a unitary operator \(U\). From this it follows easily (by considering iterates of \(U^{-1}F\)) that \(F = U\).

To prove (III), apply (II) to \(F\) as well as to \(F^{-1}\).

The following lemma contains the essence of the proof of the theorem. To state it, we introduce the notation (for \(z \in \mathbb{C}^n\))

\[
D_z = \{\lambda z : \lambda \in \mathbb{C}, \lambda z \in B\}.
\]  

(4)

Thus, when \(z \neq 0\), \(D_z\) is the disc that is the intersection with \(B\) of the complex line through 0 and \(z\).

**Lemma.** Assume that

(i) \(\Omega_1\) and \(\Omega_2\) are connected open sets in \(B\),

(ii) \(0 \in \Omega_1, 0 \in \Omega_2\),

(iii) \(F\) is a holomorphic one-to-one map of \(\Omega_1\) onto \(\Omega_2\), with \(F(0) = 0\), and

(iv) there is a nonempty open set \(V \subset \Omega_1\), such that \(D_z \subset \Omega_1\) and \(D_{F(z)} \subset \Omega_2\) for every \(z \in V\).

Then there is a unitary transformation \(U\) on \(\mathbb{C}^n\) such that \(F(z) = Uz\) for all \(z \in \Omega_1\).

**Proof of the Lemma.** If \(z \in V\), then \(D_z\) lies in the domain of \(F\). Identifying \(D_z\) with \(Bx\), we see from fact (I) (the case \(k = 1\)), that \(|w| < |z|\), where \(w = F(z)\). But \(D_w\) lies in the domain of \(F^{-1}\), and the same argument shows that \(|z| < |w|\). Thus \(|F(z)|^2 = |z|^2\) for all \(z \in V\). Both of these functions are real-analytic, hence they are equal in all of \(\Omega_1\). In particular, choosing \(r > 0\) so small that \(rB \subset \Omega_1\), we see that \(|F(z)| = |z|\) for all \(z \in rB\). An appropriately scaled version of fact (III) shows now that \(F\) is unitary.

**Proof of the Theorem.** Let \(\{a_i\}\) be as in assumption (d), put \(b_i = F(a_i)\), and choose \(u_i \in S, v_i \in S\), so that

\[
a_i = |a_i|u_i, \quad b_i = |b_i|v_i, \quad (i = 1, 2, 3, \ldots).
\]  

(5)

The geometric information contained in (d) shows that there exists \(t < 1\) such that, setting

\[
E_i(\xi) = \{z \in B : t < \text{Re}\langle z, \xi \rangle\}, \quad (\xi \in S),
\]  

(6)

we have \(a_i \in E_i(u_i) \subset \Omega_1\), and \(b_i \in E_i(v_i) \subset \Omega_2\) for all sufficiently large \(i\), say \(i > i_0\).

If \(a \in B \setminus \{0\}\), let \(P\) denote the orthogonal projection of \(\mathbb{C}^n\) onto the one-dimensional subspace spanned by \(a\), put \(Q = I - P\), and define

\[
q_a(z) = \frac{a - Pz - (1 - |a|^2)^{1/2}Qz}{1 - \langle z, a \rangle}, \quad (z \in \overline{B}).
\]  

(7)
Then (see [4], for instance) \( \varphi_a \in \text{Aut}(B) \) and \( \varphi_a^{-1} = \varphi_a \). Define

\[
G_i = \varphi_b \circ F \circ \varphi_{a_i}, \quad (i > i_0).
\]

(8)

Each \( G_i \) is a holomorphic one-to-one map of \( \Omega_i' = \varphi_a(\Omega_i) \) onto \( \Omega_2' = \varphi_b(\Omega_2) \), and \( G_i(0) = 0 \).

If \( a = |a| \xi \), then \( \langle Pz, \xi \rangle = \langle z, \xi \rangle \xi \), hence

\[
\langle \varphi_a(z), \xi \rangle = (|a| - \langle z, \xi \rangle) / (1 - |a| \langle z, \xi \rangle).
\]

(9)

If \( \tau < |a| \), it follows that \( \varphi_a(E(\xi)) \) contains all \( z \in B \) with

\[
\text{Re}\langle z, \xi \rangle < (|a| - \tau) / (1 - |a| \tau).
\]

(10)

Since \( |a_i| \to 1 \) and \( |b_i| \to 1 \), and since the right side of (10) tends to 1 as \( |a| \) tends to 1, there is a sequence \( \{r_i\} \), \( r_i < 1 \), such that \( r_i \to 1 \) as \( i \to \infty \), and such that

\[
z \in B, \text{Re}\langle z, u_i \rangle < r_i \text{ implies } z \in \Omega_i',
\]

(11)

\[
w \in B, \text{Re}\langle z, v_i \rangle < r_i \text{ implies } w \in \Omega_2'.
\]

(12)

By (11), \( r_i B \subset \Omega_i' \), the domain of \( G_i \). Since \( G_i(0) = 0 \), fact (II) gives \( |(JG_i)(0)| < r_i^{-n} \). In the same way, (12) leads to \( |(JG_i)(0)| < r_i^{-n} \), so that \( |(JG_i)(0)| > r_i^n \). A normal family argument shows now that a subsequence of \( \{G_i\} \) converges, uniformly on compact subsets of \( B \), to a holomorphic map of \( B \) into \( B \) that fixes 0 and whose Jacobian at 0 has absolute value 1. By fact (II), this limit map is unitary.

Call it \( U \).

Let \( V_i \) be the set of all \( p \in B \) such that

\[
D_z \subset \Omega_i' \quad \text{and} \quad D_w \subset \Omega_2'.
\]

(13)

for all \( z \) in some neighborhood of \( p \).

Now fix \( \varepsilon, 0 < \varepsilon < 1/10 \). Using (11)–(13), we see that there is an index \( i \), fixed from now on, such that

\[
|G_i(z) - Uz| < \varepsilon \quad \text{whenever } |z| < 1 - \varepsilon,
\]

(14)

and such that \( V_i \) contains a ball of radius \( 2 \varepsilon \), whose center \( p \) satisfies \( |p| < 1 - 3 \varepsilon \). To see in more detail that this can indeed be done, note that when \( r_i \) is sufficiently close to 1, there exists a large set of points \( \xi \in S \) such that \( \langle \xi, u_i \rangle < r_i \) and \( \langle \xi, U^{-1}v_i \rangle < r_i \). For any such \( \xi \), \( D_\xi \subset \Omega_i' \) and \( D_{U\xi} \subset \Omega_2' \), thus \( \lambda \xi \in V_i \) if \( 0 < |\lambda| < 1 \).

Thus \( D_z \subset \Omega_i' \) if \( |z - p| < 2 \varepsilon \), and \( D_w \subset \Omega_2' \) if \( |w - Up| < 2 \varepsilon \). If \( |z - p| < \varepsilon \), and \( w = G_i(z) \), it follows that \( D_w \subset \Omega_2' \) because

\[
|w - Up| < |G_i(z) - Uz| + |z - p| < 2 \varepsilon.
\]

(15)

The lemma applies therefore to \( G_i \) and shows that \( G_i \) is (the restriction of) a unitary operator. Since (8) gives

\[
F = \varphi_b \circ G_i \circ \varphi_a,
\]

(16)

the theorem is proved.

Remarks. (i) Let \( \Omega \) be a connected open subset of \( B \) such that \( \Omega \) contains an open subset \( \Gamma \) of \( S \). If \( F \) is a nonconstant \( C^1 \)-map of \( \Omega \) into \( B \) that is holomorphic in \( \Omega \) and carries \( \Gamma \) into \( S \), then \( F \in \text{Aut}(B) \). This was proved by Pinčuk [6, p. 381],
who extended an earlier version due to Alexander [1] in which $C^\infty$ was assumed in place of $C^1$.

This Alexander-Pinčuk result is a fairly direct corollary of the present theorem. If $F \in C^1(\Omega)$ satisfies the Alexander-Pinčuk hypotheses, it is not hard to show (see Fornaess [3, p. 549] or Pinčuk [6, p. 378]) that $JF$ vanishes at no point of $\Gamma$. The inverse function theorem implies then that the hypotheses of the present theorem hold.

(ii) In Alexander's proof [2] that every proper holomorphic map of $B$ into $B$ is in $\text{Aut}(B)$ when $n > 1$, his appeal to Fefferman's theorem can be replaced by the one proved in the present paper. Consequently, there exists now a much more elementary proof of the proper mapping theorem for $B$.

(iii) It is quite possible that the present theorem remains true if $B$ is replaced by strictly pseudoconvex domains with real-analytic boundaries (as Pinčuk did in the $C^1$-case [7]), but an entirely different proof would have to be found; Rosay [8] (strengthening a result of Wong [9]) proved that if some boundary point $\xi$ of a bounded domain $\Omega \subset \mathbb{C}^n$ is a point of strict pseudoconvexity, and if there exist automorphisms $T_k$ of $\Omega$ such that $\lim_{k \to \infty} T_k(p) = \xi$ for some $p \in \Omega$, then $\Omega$ is biholomorphically equivalent to $B$.

In other strictly pseudoconvex bounded domains there are thus insufficiently many automorphisms to imitate the proof that works in $B$.

(iv) If $\xi \in S$ and $\Omega = B \cap \{z: |\xi - z| < 1\}$; in other words, if $\Omega = B \cap (\xi + B)$, then the map $z \mapsto \xi - z$ of $\Omega$ onto $\Omega$ demonstrates the relevance of the assumptions concerning the location of the points $\alpha$ and $\beta$ in our theorem.

References


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