NOT EVERY \( d \)-SYMMETRIC OPERATOR IS GCR

C. RAY ROSENTRATER

ABSTRACT. Let \( T \) be an element of \( \mathcal{B}(\mathcal{H}) \), the algebra of bounded linear operators on the Hilbert space \( \mathcal{H} \). The derivation induced by \( T \) is the map \( \delta_T(X) = TX - XT \) from \( \mathcal{B}(\mathcal{H}) \) into itself. \( T \) is \( d \)-symmetric if the norm closure of the range of \( \delta_T \), \( \mathcal{R}(\delta_T) \), is closed under taking adjoints. This paper answers the question of whether every \( d \)-symmetric operator is GCR by giving an example of an NGCR weighted shift that is also \( d \)-symmetric.

Let \( \mathcal{H} \) be a complex Hilbert space and \( T \) an element of \( \mathcal{B}(\mathcal{H}) \), the algebra of bounded linear operators from \( \mathcal{H} \) into \( \mathcal{H} \). The derivation induced by \( T \) is the mapping \( \delta_T(X) = TX - XT \) from \( \mathcal{B}(\mathcal{H}) \) into itself. \( T \) is said to be \( d \)-symmetric if the norm closure of the range of \( \delta_T \), \( \mathcal{R}(\delta_T) \), is closed under taking adjoints. Examples of \( d \)-symmetric operators include the normal operators and isometries.

In [ABDW] it is proved that a necessary and sufficient condition for \( T \) to be \( d \)-symmetric is that \( \mathcal{T}T^* - T^*T \in \mathcal{C}(T) \) where \( \mathcal{C}(T) = \{ C \in \mathcal{B}(\mathcal{H}) : C \mathcal{B}(\mathcal{H}) C \subseteq \mathcal{R}(\delta_T)^- \} \). In the same paper the question is raised whether every \( d \)-symmetric operator is GCR. This paper answers that question in the negative by giving an example of a weighted shift \( T e_i = \alpha_i e_{i+1} \), \( i \in \mathbb{Z} \), that is both \( d \)-symmetric and NGCR. Recall that an operator \( T \) is GCR if every irreducible representation of \( \mathcal{C}^*(T) \), the \( \mathcal{C}^* \)-algebra generated by \( T \) and the identity operator, contains the compact operators. \( T \) is NGCR if \( \mathcal{C}^*(T) \) contains no GCR two sided ideal [A]. If \( T \) is irreducible then \( T \) is NGCR if and only if \( \mathcal{C}^*(T) \) contains no nonzero compact operator [A].

Lemma. Let \( V \) be similar to \( T \), say \( SVS^{-1} = T \). Then \( T \) is \( d \)-symmetric if and only if \( S^{-1}(T T^* - T^*T)S \in \mathcal{C}(V) \).

Proof. \( \delta_T(SXV)S^{-1} = S\delta_XVXV^{-1} = S\delta_X(X)S^{-1} \). Hence \( \mathcal{R}(\delta_T) = \mathcal{S}^* \mathcal{R}(\delta_T) \mathcal{S}^{-1} \) and it follows that \( \mathcal{C}(V) = \mathcal{S}^{-1} \mathcal{C}(T) \mathcal{S} \). Thus \( C = TT^* - T^*T \in \mathcal{C}(V) \) if and only if \( \mathcal{S}^{-1}CS \in \mathcal{C}(V) \). The lemma now follows from the result quoted above. \( \square \)

We now restrict our attention to weighted shifts. Recall that two bilateral shifts \( V e_i = \alpha_i e_{i+1} \) and \( T f_i = \beta_i f_{i+1} \) are similar if and only if there exist integer \( k \) and constant \( C \) so that \( 1/C \leq |(\alpha_k \alpha_{k+1} \cdots \alpha_{k+n})/(\beta_0 \beta_1 \cdots \beta_n)| \leq C \) uniformly for
all \( n \in \mathbb{Z} \) (see [S]). If we define \( T_0 e_i = \beta_{i-k} e_{i+1} \) then \( T_0 \) is unitarily equivalent to \( T \), \( T_0 \) is similar to \( V \), and the similarity can be implemented by an operator that is diagonal with respect to \( \{ e_n \} \) (see [S]). The same results are true in the unilateral case with \( k = 0, n \in \mathbb{N} \). This leads to the following.

**Corollary.** Let \( V \) and \( T \) be similar (unilateral or bilateral) weighted shifts. Then \( T \) is \( d \)-symmetric if and only if \( T_0 T_0^* - T_0^* T_0 \in \mathcal{C}(V) \).

**Proof.** Since \( d \)-symmetry is clearly preserved under unitary equivalence, \( T \) is \( d \)-symmetric if and only if \( T_0 \) is \( d \)-symmetric. \( T_0 \) is similar to \( V \) by means of a diagonal operator \( D \). \( T_0 T_0^* - T_0^* T_0 \) is diagonal with respect to the same basis so \( D^{-1}(T_0 T_0^* - T_0^* T_0)D = T_0 T_0^* - T_0^* T_0 \). \( \square \)

**Remark.** If \( S \) is an invertible operator that commutes with \( C \), then it is not hard to show that \( C \in \mathcal{C}(T) \) if and only if \( CS \in \mathcal{C}(T) \). In particular, if \( C = T_0 T_0^* - T_0^* T_0 \) is a diagonal operator as in the corollary, then \( C \in \mathcal{C}(T) \) if and only if \( |C| \), the diagonal with diagonal entries the modulus of the corresponding entry in \( C \), is in \( \mathcal{C}(T) \). (It is not true in general that \( |C| \in \mathcal{C}(T) \) implies \( C \in \mathcal{C}(T) \).)

In [ABDW] it is shown that when \( T \) is \( d \)-symmetric, \( \mathcal{C}(T) \) is the linear span of the positive elements in \( \Re(\delta_T)^- \). This implies the following.

**Proposition.** If \( V \) is a \( d \)-symmetric weighted shift and \( T \) is a weighted shift similar to \( V \), then \( T \) is \( d \)-symmetric if and only if \( |T_0 T_0^* - T_0^* T_0| \in \Re(\delta_T)^- \).

Before we proceed to the example, we need to state a result due to O'Donovan. In [O] he proves that a bilateral shift with nonzero weights \( \{ w(i) \} \) is NGCR if and only if there exists a sequence \( n_k \to \infty \), such that \( w(i + n_k) \to w(i) \) for \( i \in \mathbb{Z} \).

**Example.** Let \( T \) be the bilateral weighted shift with weights defined by

\[
\begin{align*}
w(i) &= \begin{cases}
1, & i < 0, \\
\frac{1}{2}, & i = 1, \\
2, & i = 2, \\
1, & 3^k < i < 2 \cdot 3^k, \\
w(i - 2 \cdot 3^k), & 2 \cdot 3^k < i < 3^k + 1.
\end{cases}
\end{align*}
\]

**Claim I.** \( T \) is NGCR.

**Proof.** Let \( n_k = 2 \cdot 3^k \). Fix \( i < 0 \). Then for \( k > 1 \) so that \( 3^k > |i| \), \( w(i + n_k) = w(2 \cdot 3^k - |i|) = 1 = w(i) \).

Fix \( i > 0 \). Then for \( k \) so that \( 3^k > i \) we have \( 2 \cdot 3^k < i + 2 \cdot 3^k < 3^k + 1 \) so \( w(i + n_k) = w(i + 2 \cdot 3^k) = w(i) \). In any case we have \( w(i + n_k) \to w(i) \). \( \square \)

**Claim II.** \( T \) is similar to the bilateral shift \( V_{e_n} = e_{n+1} \) and \( T_0 = T \).

**Proof.** An induction argument shows that if \( w(k) = 2 \) then \( w(k - 1) = \frac{1}{2} \) and if \( w(k) = \frac{1}{2} \) then \( w(k + 1) = 2 \). Since all other weights are 1 it follows that

\[
\frac{1}{2} < |w(0) \cdot w(1) \cdot \cdots \cdot w(n)| < 2 \quad \text{for } n \in \mathbb{Z}. \quad \square
\]
Matrix computations show that \( D = |TT^* - T^*T| \) is the diagonal operator with the weights \[
d(i) = \begin{cases} 
0, & i < 0, \\
\frac{1}{4}, & i = 1, \\
\frac{15}{4}, & i = 2, \\
3, & i = 3, \\
0, & 3^k < i < 2 \cdot 3^k, \\
d(i - 2 \cdot 3^k), & 2 \cdot 3^k < i < 3^{k+1}.
\end{cases}
\]

In order to show \( T \) is \( d \)-symmetric it is enough to show that \( D = |TT^* - T^*T| \in \mathcal{R}(\delta_{\nu})^{-} \) by the proposition. As \[
\delta_{\nu} \left( - \sum_{j=0}^{n-1} \left( \frac{n-j}{n} \right) V^j D V^{*j(j+1)} \right) = D - \frac{1}{n} \sum_{j=1}^{n} V^j D V^{*j},
\]
we will be done if we show \( 3^k \| \sum_{j=1}^{3^k} V^j D V^{*j} \| \to 0 \) as \( k \to \infty \).

Since conjugation by \( V \) shifts a diagonal operator one position down the diagonal, \( \sum_{j=1}^{3^k} V^j D V^{*j} \) is also a diagonal operator and its weights are \( d'(i) = \sum_{j=1}^{3^k} d(i - j) = \sum_{j=1}^{3^k} d(i - n + j) \). Thus it suffices to show that \[
\frac{1}{3^k} \sum_{j=1}^{3^k} d(i + j) \to 0 \quad \text{uniformly in } i \text{ as } k \to \infty.
\]

**Claim III.** \( \sum_{j=1}^{3^k} d(j) < 8 \cdot 2^{k} \).

**Proof.** If \( k = 1 \), then \( \sum_{j=1}^{3} d(j) = 15/2 < 8 \). Assuming \( \sum_{j=1}^{3^k} d(j) < 8 \cdot 2^{k} \) we see that \[
\sum_{j=1}^{3^{k+1}} d(j) = \sum_{j=1}^{3^k} d(j) + \sum_{j=3^k+1}^{2 \cdot 3^k} d(j) + \sum_{j=2 \cdot 3^k+1}^{3^{k+1}} d(j)
\]
\[
= 2 \sum_{j=1}^{3^k} d(j) < 8 \cdot 2^{k+1}.
\]

**Claim IV.** \( \sum_{j=1}^{3} d(i + j) < 8 \cdot 2^{l} \) for all \( i \in \mathbb{Z} \).

**Proof.** Suppose that \( -\infty < i < 3^{l} \). Since \( d(j) = 0 \) for \( j < 0 \) and \( 3^{l} < j < 2 \cdot 3^{l} \), \[
\sum_{j=1}^{3^{l}} d(i + j) = \sum_{j=i+1}^{3^{l}} d(j) < \sum_{j=1}^{2 \cdot 3^{l}} d(j)
\]
\[
= \sum_{j=1}^{3^{l}} d(j) < 8 \cdot 2^{l}.
\]
by Claim III.

Let \( k > l \) and assume that \( \sum_{j=1}^{3^{l}} d(i + j) < 8 \cdot 2^{l} \) for \( i < 3^{k} \). Let \( 3^{k} < i < 3^{k+1} \) and consider \[
\sum_{j=1}^{3^{l}} d(i + j) = \sum_{j=i+1}^{i+3^{l}} d(j).
\]
If \( i + 3^l < 2 \cdot 3^k \) then the sum is zero since \( d(j) = 0 \) for \( 3^k < j < 2 \cdot 3^k \). For the same reason we can assume that the lower limit on the sum is at least \( 2 \cdot 3^k \). Since \( d(j) = 0 \) for \( 3^k+1 < j < 3^k+1 + 3^l < 2 \cdot 3^k+1 \), we can also assume that the upper limit is at most \( 3^k+1 \). Hence \( \sum_{j=n+1}^{m} d(j) = \sum_{j=n+1}^{m+3^l} d(j) \) with \( 2 \cdot 3^k < n < m < 3^k+1 \) and \( m - n < 3^l \). Let \( m' = m - 2 \cdot 3^k \) and \( n' = n - 2 \cdot 3^k \); then we see that
\[
\sum_{j=n+1}^{m} d(j) = \sum_{j=n'+1}^{m'} d(j) < \sum_{j=n'+1}^{n'+3^l} d(j) < 8 \cdot 2^l
\]
since \( n' < 3^k+1 - 2 \cdot 3^k = 3^l \). Hence
\[
\sum_{j=1}^{3^l} (i + j) < 8 \cdot 2^l \text{ for } i < 3^k+1. \quad \square
\]

Thus we have shown that \( T \) is both NGCR and \( d \)-symmetric.

**References**


**Department of Mathematics, Indiana University, Bloomington, Indiana 47401**

**Current address**: Department of Mathematics, Westmont College, Santa Barbara, California 93108