AN INTEGRAL FORMULA ON THE SCALAR CURVATURE OF ALGEBRAIC MANIFOLDS

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Abstract. It is proved in this note that if the scalar curvature of an \( n \)-dimensional algebraic complex submanifold is bigger than \( n^2 \), then it is totally geodesic.

0. Introduction. Let \( \rho \) denote the scalar curvature of an \( n \)-dimensional compact connected Kaehlerian submanifold \( M \) in a complex projective space with the standard Fubini-Study metric of constant holomorphic sectional curvature 1, denoted by \( \mathbb{CP}^{n+p}(\mathbb{C}) \) where \( p \) is the codimension. And, let \( \mathbb{Q}^n \) denote the hyperquadric which is defined by

\[
\{ z = [z_0, z_1, \ldots, z_{n+1}] \in \mathbb{CP}^{n+1}(\mathbb{C}) \mid z_0^2 + z_1^2 + \cdots + z_{n+1}^2 = 0 \}
\]

with the induced metric. In this note, we shall prove the following result.

Theorem. With the above notations, we have

\[
\int_M [\rho - n^2] [\rho - n(n + 1)] \ast 1 > 0
\]

where \( \ast \) denotes the Hodge star operator and the equality holds if and only if \( M \) is holomorphically isometric to either \( \mathbb{CP}^n(\mathbb{C}) \) or \( \mathbb{Q}^n \).

Let \( \| \sigma \| \) denote the length of the second fundamental form \( \sigma \) of \( M \) embedded in \( \mathbb{CP}^{n+p}(\mathbb{C}) \), then we have the following corollary.

Corollary 1. \( \int_M \| \sigma \|^2 (\| \sigma \|^2 - n) \ast 1 > 0 \) and the equality holds if and only if \( \| \sigma \|^2 \) is identically equal to either 0 or \( n \) and \( M \) is holomorphically isometric to either \( \mathbb{CP}^n(\mathbb{C}) \) or \( \mathbb{Q}^n \) respectively.

Proof. Observe that \( \rho = n(n + 1) - \| \sigma \|^2 \) (cf. [4]).

Corollary 2. If \( \rho > n^2 \), then \( M \) is totally geodesic.

Proof. Note that \( \rho \) is not bigger than \( n(n + 1) \). Also see [4].

Remark. In [4], Corollary 2 was conjectured by K. Ogiue even in the case where \( M \) is complete.

1. Preliminaries. Let \( M \) be a Kaehler manifold of complex dimension \( n \) and \( (\theta^1, \ldots, \theta^n) \) form a local field of unitary coframes, then the Kaehler metric \( g \) and
the associated Kaehler form $\phi$ are given respectively by
\[
g = \frac{1}{2} \sum (\theta^i \otimes \bar{\theta}^i + \bar{\theta}^i \otimes \theta^i),
\]
\[
\phi = \frac{\sqrt{-1}}{2} \sum \theta^i \wedge \bar{\theta}^i.
\]
Let $\Theta^i_j = \sum R^i_{jk}\theta^k \wedge \bar{\theta}^l$ be the curvature form of $M$. The Ricci tensor $S$ and the scalar curvature $\rho$ are given respectively by
\[
S = \frac{1}{2} \sum \left( R^i_j \theta^i \otimes \theta^j + \frac{1}{n} R^i_j \theta^i \otimes \theta^j \right) \quad \text{and} \quad \rho = 2 \sum R^i_i
\]
where $R^i_j = 2 \sum R^k_{ij}$. Put $\gamma = (\sqrt{-1}/4\pi) \sum R^i_j \theta^i \wedge \theta^j$. Then its cohomology class $[\gamma]$ is the first Chern class $c_1$ (or $c_1(M)$) of $M$.

We define the $k$th scalar curvature $\rho_k$ by $\rho_k = ((2\pi)^k/n!) \ast (\phi^{n-k} \wedge \gamma^k)$. It is easy to check $\rho_0 = 1$, $\rho_1 = (1/2n)\rho$, $\rho_2 = (1/4n(n-1))(\rho^2 - 2\|S\|^2)$ where $\|S\|$ denotes the length of the Ricci tensor $S$.

We need the following facts.

**Fact 1** (B. Y. Chen [1]). Let $M$ be an $n$-dimensional compact Kaehler manifold. Then
\[
\int_M \rho_k \ast 1 = \frac{(4\pi)^n}{n!2^k} \binom{n}{k} \left[ M \right]
\]
where $\Omega$ denotes the normalized Kaehler class $(1/4\pi)[\phi]$.

**Fact 2** (B. Y. Chen and K. Ogiue [2]). Let $M$ be an $n$-dimensional Kaehler manifold. Then $2\pi\|S\|^2 > \rho^2$ and the equality holds if and only if $M$ is Einsteinian.

Now, we also need the following basic result in algebraic geometry.

**Fact 3** (A. J. Sommese [6] or A. Van de Ven [7]). Let $L$ be a very ample line bundle on an algebraic surface $X$, and let $K_X$ be the canonical bundle of $X$. Then $K_X \otimes L$ is spanned by its sections if and only if $L$ is not one of the following:

(i) $O(1)$ or $O(2)$ on $CP^2$ where $O(1)$ denotes the hyperplane section bundle and $O(2) = O(1) \otimes O(1)$,

(ii) a line bundle on a ruled surface, whose restriction to any fibre $CP^1$ is $O(1)$.

2. **Proof of the Theorem.** Straightforward computations give
\[
[\rho - n^2][\rho - n(n + 1)] = \rho^2 - (2n^2 + n)\rho + n^3(n + 1) \quad (A)
\]
and
\[
\int_M \rho \ast 1 = \int_M 2n\rho_1 \ast 1 = \frac{(4\pi)^n}{n!2^{n-1}} \binom{n}{n-1} \left[ M \right]. \quad (B)
\]

Since $\rho_2 = (1/4n(n-1))(\rho^2 - 2\|S\|^2)$, by combining Facts 1 and 2, we get the following estimation of the integral of $\rho^2$
\[
\int_M \rho^2 \ast 1 > \frac{(4\pi)^n}{n!2^{n-2}} \binom{n}{n-2} \left[ M \right] \quad (C)
\]
and equality holds if and only if \( M \) is Einsteinian. Now putting (A), (B), and (C) together, we have

\[
\int_M \left[ \rho - n^2 \right] \left[ \rho - n(n + 1) \right] \cdot 1
\]

\[
> \frac{(4\pi)^n \cdot n}{(n - 1)!} \left( [c_1 - n\Omega] [c_1 - (n + 1)\Omega] \Omega^{n-2} \right) [M]. \quad (D)
\]

We shall show that the right-hand side of (D) is nonnegative after a lemma in algebraic geometry.

Let \( L \) be the hyperplane section bundle over \( M \), i.e. the pullback of \( O_{CP^n}(1) \). Then \( L \) is very ample.

**LEMMA.** \( K_M \otimes L^n \) is spanned unless \( M \) is a linear subspace \( CP^n \) of \( CP^{n+p} \).

**PROOF.** Suppose \( K_M \otimes L^n \) is not spanned. Observe \( (K_M \otimes L)|_{M \cap H} = K_{M \cap H} \) by the adjunction formula, where \( H \) is a generic hyperplane of \( CP^{n+p} \). Then we see that \( (K_M \otimes L^n)|_X = K_X \otimes L^2_X \) is not spanned either for some \( X = M \cap H_1 \cap \cdots \cap H_{n-2} \) where \( H_i \)'s are generic hyperplanes of \( CP^{n+p} \), \( 1 < i < n - 2 \), and \( L_X \) denotes the restriction of \( L \) to \( X \). Now by Fact 3, there are two cases to be discussed. Case (i). \( X \) is biholomorphic to \( CP^2 \) and \( L^2_X \) is either \( O(1) \) or \( O(2) \).

Clearly \( L^2_X \) cannot be \( O(l) \) since every line bundle over \( CP^n \) has to be \( O(l) \) for some integer \( l \). Thus \( L^2_X = O(2) \) and \( L_X = O(1) \) which implies that the degree of \( M \) in \( CP^{n+p} \) is equal to 1 and \( M \) has to be a linear subspace \( CP^n \) of \( CP^{n+p} \). The case (ii) in Fact 3 is impossible because \( L^2_X|_{CP^1} \) cannot be \( O_{CP^1}(1) \) for the same reason as above. So we have shown our lemma.

Now, if \( M \) is a linear \( CP^n \), then the right-hand side of (D) vanishes since \( c_1(CP^n) = (n + 1)\Omega \). And that \( K_M \otimes L^n \) is spanned tells us \( K_M \otimes L^{n+1} \) is very ample since \( L \) is very ample. Hence

\[
c_1(K_M \otimes L^n)c_1(K_M \otimes L^{n+1}) = \left[ -c_1(M) + nc_1(L) \right] \left[ -c_1(M) + (n + 1)c_1(L) \right] = \left[ c_1(M) - n\Omega \right] \left[ c_1(M) - (n + 1)\Omega \right]
\]

is nonnegative which implies the right-hand side of (D) is also nonnegative. So we have proved the inequality in the theorem.

Now, if the equality in our integral formula holds, then

\[
0 = ([n\Omega - c_1] [(n + 1)\Omega c_1] \Omega^{n-2}) [M]
\]

which forces \( n\Omega = c_1 \) unless \( M \) is a linear \( CP^n \) by using the following fact: if \( \phi \) is a nonnegative (1, 1) form and \( \psi \) is a positive \((n - 1, n - 1)\) form, then \( \int_M \phi \wedge \psi = 0 \) if and only if \( \phi = 0 \). On the other hand, \( M \) is Einsteinian since the equality of (c) holds. Now either \( M \) is a linear \( CP^n \) i.e. \( M \) is holomorphically isometric to \( CP^n(1) \) or \( c_1 = n\Omega \) which implies \( M \) is biholomorphic to \( Q^n \) as a hypersurface of a linear subspace \( CP^{n+1}(1) \) of \( CP^{n+p}(1) \) (S. Kobayashi and T. Ochiai [3]). And since \( M \) is Einsteinian, \( M \) has to be also isometric to \( Q^n \) (B. Smyth [5]). Conversely, either \( CP^n(1) \) or \( Q^n \) surely makes the equality in our integral formula hold. So we complete the proof.
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Appendix. For $n = 1$, we estimate the integral of $\rho^2$ by the Schwarz inequality.

\[
\left( \int_M \rho^2 \star 1 \right) \left( \int_M \star 1 \right) > \left( \int_M \rho \star 1 \right)^2.
\]

Let $A = \int_M \rho \star 1$ and $v = \int_M \star 1$. Then $A = 4\pi c_1[M] = 4\pi(2 - 2g) < 8\pi$ where $g$ is the genus of $M$, and $v = 4\pi d$ where $d$ is the degree of $M$. Hence

\[
\int_M (\rho - 1)(\rho - 2) \star 1 > v \left( \frac{A}{v} - 2 \right) \left( \frac{A}{v} - 1 \right) > 4\pi d \left( \frac{2}{d} - 2 \right) \left( \frac{2}{d} - 1 \right) > 0
\]

and the equality holds if and only if $\rho$ is constant and equal to either 1 or 2, which implies that $M$ is holomorphically isometric to either $Q^1$ or $CP^1(1)$ respectively.

References

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