EGOROFF'S THEOREM AND THE DISTRIBUTION OF STANDARD POINTS IN A NONSTANDARD MODEL

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ABSTRACT. We study the relationship between the Loeb measure $\theta(\mu)$ of a set $E$ and the $\mu$-measure of the set $S(E) = \{x^* | x \in E\}$ of standard points in $E$. If $E$ is in the $\sigma$-algebra generated by the standard sets, then $\theta(\mu(E)) = \mu(S(E))$. This is used to give a short nonstandard proof of Egoroff's Theorem. If $E$ is an internal, * measurable set, then in general there is no relationship between the measures of $S(E)$ and $E$. However, if *$X$ is an ultrapower constructed using a minimal ultrafilter on $\omega$, then $\mu(E) \approx 0$ implies that $S(E)$ is a $\mu$-null set. If, in addition, $\mu$ is a Borel measure on a compact metric space and $E$ is a Loeb measurable set, then

$$\mu(S(E)) < \theta(\mu)(E) < \bar{\mu}(S(E))$$

where $\mu$ and $\bar{\mu}$ are the inner and outer measures for $\mu$.

The work in this paper was originally stimulated by the search for an illuminating nonstandard proof of Egoroff's Theorem. Despite the importance of such a proof it has been surprisingly elusive (see, for example, [8] or [11]). §1 of this paper presents a short, natural proof of Egoroff's Theorem using a result from §II on the distribution of standard points in a nonstandard model. The work in §II is of independent interest.

Throughout this paper $(X, \mathcal{F}, \mu)$ will denote a (standard) positive measure space with $\mu(X)$ finite; $\mathcal{M}$ will denote a standard higher order model of $X$ along with the real numbers, $\mathbb{R}$; and *$\mathcal{M}$ will denote a proper nonstandard extension of $\mathcal{M}$. We will always assume *$\mathcal{M}$ is $\aleph_1$-saturated, but any further assumptions will be explicitly stated. If $P$ is an entity in $\mathcal{M}$, *$P$ will denote the corresponding entity in *$\mathcal{M}$. Thus, in particular, *$\mu$: *$\mathcal{F} \rightarrow [0, \infty)$ denotes the extension in *$\mathcal{M}$ of $\mu$ to the * measurable sets. We use the usual notation $St(x)$ for the standard part of a finite nonstandard real and $x \approx y$ for $x$ infinitely close to $y$.

I. Egoroff's Theorem. Suppose $f_1, f_2, \ldots$ is a standard sequence of measurable functions $X \rightarrow \mathbb{R}$ and $f$: $X \rightarrow \mathbb{R}$ is also a measurable function. Egoroff's Theorem [3] states

I.1 EGOROFF'S THEOREM. If $f_n \rightarrow f$ pointwise almost everywhere then for every $\varepsilon > 0$ there is a set $A \in \mathcal{F}$ such that $\mu(A) < \varepsilon$ and $f_n \rightarrow f$ uniformly on $X \setminus A$. 

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If \( \{f_n\} \) satisfies the conclusion of Egoroff’s Theorem we will say \( \{f_n\} \) converges to \( f \) nearly uniformly. Note that Egoroff’s Theorem is false without the assumption that \( \mu(X) \) is finite.

The following characterization of nearly uniform convergence is essentially due to Robinson [8].

1.2 Definition. Suppose \( f, f_1, f_2, \ldots \) are standard measurable functions \( X \to \mathbb{R} \), and \( x \in \ast X \). Then \( x \) is said to be a point of intrinsic nonuniformity if there is an infinite integer \( v \) such that \( f_v(x) \approx \ast f(x) \). Let \( E \) denote the set of points of intrinsic nonuniformity. (Note: \( E \) is usually external.)

1.3 Definition. Suppose \( A \) is a (possibly external) subset of \( \ast X \). \( A \) is said to have \( S \)-measure zero if for every standard \( \varepsilon > 0 \) there is a standard set \( B \in \mathcal{S} \) such that \( A \subseteq B \) and \( \mu(B) < \varepsilon \).

1.4 Proposition. Using the notation of Definition 1.2, the following are equivalent.

(i) \( \{f_n\} \) converges to \( f \) nearly uniformly.

(ii) \( E \) has \( S \)-measure zero.

Proof. The proof is completely straightforward using the well-known fact that \( f_n \to f \) uniformly on a set \( S \) if and only if for every \( p \in \ast S \) and every infinite \( v \), \( f_v(p) \approx \ast f(p) \) [8, Theorem 4.6.1].

We need one more definition before proving Egoroff’s Theorem.

1.5 Definition. Suppose \( A \) is a (possibly external) subset of \( \ast X \). Let \( S(A) \) denote the set of all standard points in \( A \). That is, \( S(A) = \{x \in X | \ast x \in A\} \). Note \( S(A) \) is just the standard part of \( A \) with respect to the discrete topology on \( X \).

1.6 Proof of Egoroff’s Theorem. Suppose \( f_n \to f \) pointwise almost everywhere. Hence there is a set \( A \in \mathcal{S} \) such that \( \mu(A) = 0 \) and \( f_n \to f \) pointwise on \( X \setminus A \). Let \( E \) denote the set of points of intrinsic nonuniformity. Then \( S(E) \subseteq A \). Thus \( S(E) \) has measure zero and by II.3 \( E \) has \( S \)-measure zero, completing the proof by 1.4.

II. The distribution of standard points in \( \ast X \). The purpose of this section is to study the relationship between the measure (in a sense to be defined below) of a set \( E \subseteq \ast X \) and the standard measure of \( S(E) \). Intuitively, the standard points are evenly distributed in \( \ast X \) and one might, therefore, expect the measures of \( E \) and \( S(E) \) to be infinitely close for a reasonable class of sets \( E \).

II.1 Definition. Let \( \mathcal{Q} \) be the (external) algebra, \( \mathcal{Q} = \{\ast A | A \in \mathcal{F} \} \), and let \( \mathcal{S} \) be the (external) \( \sigma \)-algebra generated by \( \mathcal{Q} \). Using the Loeb-Carathéodory extension process there is an (external) real-valued \( \sigma \)-additive measure \( \mathcal{Q}(\mu): \mathcal{S} \to [0, \infty) \) [5], see also [8, §5.1], called \( \mathcal{S} \)-measure. Notice \( \mathcal{Q}(\mu)(A) = 0 \) if and only if \( A \) has \( S \)-measure zero in the sense of 1.3.

II.2 Theorem. Suppose \( E \in \mathcal{S} \). Then \( S(E) \in \mathcal{F} \) and \( \mathcal{Q}(\mu)(E) = \mu(S(E)) \).

Proof. First, let \( \mathcal{T}_1 = \{E \in \mathcal{S} | S(E) \in \mathcal{F} \} \). \( \mathcal{T}_1 \) is a \( \sigma \)-algebra since \( S(\ast X \setminus A) = X \setminus S(A) \), \( S(A_1 \cap A_2) = S(A_1) \cap S(A_2) \) and \( S(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} S(A_n) \). Hence \( \mathcal{S} = \mathcal{T}_1 \). Now we have two finite measures defined on \( \mathcal{S} \), \( \mu_1(E) = \mathcal{Q}(\mu)(E) \) and \( \mu_2(E) = \mu(S(E)) \). By the uniqueness part of the Caratheodory Extension Theorem, we have \( \mu_1 = \mu_2 \) completing the proof. Notice the importance here that \( \mu(X) \) is finite.
11.3 Example. Let $E$ be as in 1.6, the set of points of intrinsic nonuniformity for $f$ and $(f_n)$ where $f_n \to f$ almost everywhere. Then $E \in \mathcal{S}$ and $\mathcal{O}(\mathcal{\mu})(E) = \mu(S(E)) = 0$.

Proof. Let $A_{n,k} = \{ x \in X \mid \exists r > k |f_r(x) - f(x)| > 1/n \}$. Claim:

$$E = \bigcup_{n=1}^{\infty} \bigcap_{k=1}^{\infty} A_{n,k}.$$  

Proof of claim. If $x \in E$ then there is an infinite $\rho$ such that $f_\rho(x) \approx \ast f(x)$. Therefore, there is a finite $n$ such that $|f_\rho(x) - f(x)| > 1/n$. Thus $x \in A_{n,\rho}$ for every finite $k$. \ldots$x \in \bigcap_{k=1}^{\infty} A_{n,k}$. Conversely, suppose $x \in \bigcup_{n=1}^{\infty} \bigcap_{k=1}^{\infty} A_{n,k}$. Therefore there is an $n$ such that $x \in \bigcap_{k=1}^{\infty} A_{n,k}$. Let $T = \{ k \mid |f_k(x) - \ast f(x)| > 1/n \}$. $T$ is internal and contains arbitrarily large finite positive integers $k$. Therefore $T$ contains some infinite positive integer $\nu$ and $|f_\nu(x) - \ast f(x)| > 1/n$. So $x \in E$.

One of the difficulties in applying the techniques of Nonstandard Analysis to standard problems is converting a nonstandard object into a standard one. In particular, if $F: \ast X \to \mathbb{R}$ is Loeb measurable and we define $f: X \to \mathbb{R}$ by $f(x) = F(x)$ then we have very little control over $f$. In fact, $f$ need not be measurable and $\ast f$ need have little relationship to $F$. One consequence of Theorem II.2 is that the situation is much better if $F$ is $\mathcal{S}$-measurable. More precisely, we have

11.4 Theorem. Suppose $F: \ast X \to \mathbb{R}$ is $\mathcal{S}$-measurable and $f: X \to \mathbb{R}$ is defined by $f(x) = F(x)$. Then

1. $f$ is $\mathcal{S}$-measurable,
2. \{ $x \in \ast X | \ast f(x) \approx F(x)$ \} has $\mathcal{S}$-measure zero.

Proof. (1) Let $t \in \mathbb{R}$ and $A = \{ x \mid f(x) > t \}$. Notice $A = S(E)$ where $E = \{ x \mid F(x) > t \} \in \mathcal{S}$ by assumption. Hence, by Theorem II.2 $A \in \mathcal{S}$ and $f$ is $\mathcal{S}$-measurable.

(2) Let $0(\ast f): \ast X \cup \{ \infty \}$ be given by

$$0(\ast f)(x) = \begin{cases} \text{St}(\ast f(x)) & \text{if } \ast f(x) \text{ is finite}, \\ \infty & \text{if } \ast f(x) \text{ is infinite}. \end{cases}$$

A straightforward argument shows $0(\ast f)$ is $\mathcal{S}$-measurable. Let $E = \{ x \mid 0(\ast f)(x) \neq F(x) \}$. Then $E \in \mathcal{S}$ and $S(E) = \emptyset$. So, $E$ has $\mathcal{S}$-measure zero by Theorem II.2. But $E = \{ x \mid \ast f(x) \approx F(x) \}$ completing the proof.

The obvious question to ask is whether Theorem II.2 can be extended to a larger class of sets. A natural such question is whether for internal $\ast$measurable sets $E$, $\ast \mu(E) \approx \mu(S(E))$. Unfortunately, the possible results in this direction are sharply circumscribed by the following examples.

11.5 Example. Suppose $X = [0, 1]$ and $\ast X$ is an enlargement of $X$. Then for every $B \subseteq [0, 1]$ and $t \in [0, 1]$ there is an internal, $\ast$Borel set $E$ such that $S(E) = B$ and $\ast \mu(E) = t$. 

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Proof. A straightforward enlargement argument produces *finite sets $F_1$, $F_2$ such that $S(F_1) = B$, $S(F_2) = [0, 1] \setminus B$ and $F_1 \cap F_2 = \emptyset$. Let
\[ E = ([0, t] \cup F_1) \setminus F_2. \]

II.6 Example. Suppose $X = \{0, 1\}^\infty$ (i.e. an element $x \in X$ is a sequence $x = (x_1, x_2, \ldots)$ of 0's and 1's). Let $X$ have the obvious probability measure. Let *$X$ be any proper nonstandard extension of $X$ and let $\nu$ be any infinite positive integer. Let $E = \{x \in *X | x_* = 1\}$. It is well known and easy to prove using the Kolmogorov Zero-One Law that $S(E)$ has inner measure zero and outer measure one. But $\mu(E) = 1/2$.

II.7 Example. We construct a nonstandard model *$\mathcal{M} = \mathcal{M}^J / D$, where $J$ is countable, such that there is an internal, *open set $V \subseteq *[0, 1]$ with $S(V) = \emptyset$ but $\mu(V) \approx 1$ ($\mu$ is Lebesgue measure). Let $X = [0, 1]$ and let $J$ be the set of all finite unions of disjoint open intervals with rational endpoints. Thus a typical element $U$ of $J$ is a set $\bigcup_{n=1}^k (a_n, b_n)$ with $a_1 < b_1 < a_2 < \cdots < b_n$ all rational. Notice $J$ is countable. If $x_1, x_2, \ldots, x_k \in [0, 1]$ let $\mathcal{U}(x_1, x_2, \ldots, x_k) = \{U \in J | x_1, x_2, \ldots, x_k \not\in U, \mu(U) > 1 - 1/k\}$. Clearly the sets $\mathcal{U}(x_1, x_2, \ldots, x_k)$ are nonempty and have the finite intersection property. Let $D$ be any ultrafilter containing all the sets $\mathcal{U}(x_1, x_2, \ldots, x_k)$. Let *$\mathcal{M} = \mathcal{M}^J / D$ and let $V \subseteq *X$ be the *open set represented by the function $F: J \to P(X)$ given by $F(U) = U$. It is immediate from the construction of $V$ that $\mu(V) \approx 1$ and $S(V) = \emptyset$.

Thus, in general, there is no relationship between $\mu(E)$ and $\mu(S(E))$. However, if *$X$ is a minimal nonstandard model (defined below) we do have some positive results.

II.8 Definition. Suppose $J = \{1, 2, 3, \ldots\}$ and $D$ is an ultrafilter on $J$. $D$ is said to be minimal, see [1], [9], [10], provided whenever $f: J \to J$ there is a set $A \in D$ such that either $f$ is constant on $A$ or $f$ is one-to-one on $A$. If either the Continuum Hypothesis or Martin’s Axiom holds there are many minimal ultrafilters on $J$ [1], [9], [10]. If $D$ is a minimal ultrafilter on $J$, the nonstandard model *$\mathcal{M} / D$ is said to be minimal. We use below the fact that minimal ultrafilters are Ramsey [2] and therefore satisfy the strong Ramsey theorem proved by Mathias [6], [7].

II.9 Proposition. Suppose *$\mathcal{M}$ is a minimal nonstandard model and $E \subseteq *X$ is an internal *measurable set such that $\mu(E) \approx 0$. Then $S(E)$ is a $\mu$-null set.

Proof. Let $e = \mu(E)$, let $(E_1, E_2, \ldots)$ represent $E$ and let $e_n = \mu(E_n)$. Hence, $e$ is represented by $(e_1, e_2, \ldots)$. Define $f: J \to J$ by $f(n) =$ largest $k$ such that $e_k < 1/2^k$. Since $D$ is minimal there is a set $A \in D$ such that $f$ is constant on $A$ or $f$ is one-to-one on $A$. Since $e \approx 0$ the first alternative is impossible. Hence $f$ is one-to-one on $A$. Therefore

\[ \sum_{n \in A} e_n < \sum_{k=1}^\infty 1/2^k = 1. \]

Now, suppose $\varepsilon > 0$ is standard. Then there is a set $B \in D$ such that

\[ \sum_{n \in B} e_n < \varepsilon. \]
But $x \in S(E)$ implies $x \in \bigcup_{n \in B} E_n$. Thus $S(E) \subseteq \bigcup_{n \in B} E_n$ but $\mu(\bigcup_{n \in B} E_n) < \Sigma_{n \in B} e_n < \epsilon$. This completes the proof.

If $\ast \mathcal{M}$ is a minimal nonstandard model, $\mu$ is a Borel measure on a compact metric space and $E$ is Loeb measurable then considerably more can be said about the relationship between the measures of $E$ and $S(E)$. The first step is the following lemma.

II.10 Lemma. Suppose $\ast \mathcal{M}$ is minimal nonstandard model, $\mu$ is a Borel measure on a compact metric space $K$ and $E$ is a Loeb measurable set such that $S(E) = \emptyset$. Then $0(\ast \mu(E)) = 0$.

Proof. Suppose $E$ is Loeb measurable, $S(E) = \emptyset$ and $0(\ast \mu(E)) > 0$. Then there is a standard $\epsilon > 0$ and an internal, $\ast$-measurable set $F \subseteq E$ such that $\ast \mu(F) > \epsilon$.

Let $(F_n)$ be a sequence of Borel subsets of $K$ which determines $F$ as an element of $\ast \mathcal{M}$. We then have that the set $Y = \{n \mid \mu(F_n) > \epsilon \}$ is an element of the minimal ultrafilter $D$ which is used to construct $\ast \mathcal{M}$. The Ramsey Theorem for $D$ due to Mathias [6] will be used to show that there exists $Z \in D$ with $Z \subseteq Y$ and

$$\cap \{F_n \mid n \in Z\} \neq \emptyset.$$

Since this intersection is contained in $S(F)$, this contradicts our assumption that $S(F) = \emptyset$.

Given an infinite set $W \subseteq N$, set $[W]^\omega = \{V \mid V$ is an infinite subset of $W\}$. On $[N]^\omega$ put the usual topology: the basic open neighborhoods of $W \in [N]^\omega$ are the sets

$$\{V \in [N]^\omega \forall k < n (k \in V \iff k \in W)\}$$

for $n = 1, 2, \ldots$. The Ramsey Theorem of Mathias implies that if $\mathcal{R} \subseteq [N]^\omega$ is analytic relative to this topology, there exists some $W \in D$ such that either $[W]^\omega \subseteq \mathcal{R}$ or $[W]^\omega \cap \mathcal{R} = \emptyset$. (By [2] a minimal ultrafilter is Ramsey; for a proof of Mathias' result that Ramsey ultrafilters have this much stronger property see [7].

For our purposes we use the family

$$\mathcal{R} = \{W \in [N]^\omega \mid \cap \{F_n \mid n \in W\} \neq \emptyset\}.$$

First we show that $\mathcal{R}$ is analytic. Consider the set $S \subseteq [N]^\omega \times K$ defined by

$$S = \{(W, x) \mid x \in \cap \{F_n \mid n \in W\}\}.$$

Since each $F_n$ is a Borel subset of $K$, $S$ is a Borel set in the product space $[N]^\omega \times K$. Also $\mathcal{R}$ is the image of $S$ under the coordinate projection from $[N]^\omega \times K$ onto $[N]^\omega$. Since $K$ is a compact metric space, it follows that $\mathcal{R}$ is analytic in $[N]^\omega$ [4, Chapter XIII].

Now apply Mathias' theorem, obtaining a set $W \in D$ such that $[W]^\omega \subseteq \mathcal{R}$ or $[W]^\omega \cap \mathcal{R} = \emptyset$. In the first case we have $Z = Y \cap W \in D$ and $\cap \{F_n \mid n \in Z\} \neq \emptyset$ as desired. Thus it suffices to prove the second case is impossible. For any $W \in D$ the set $Z = Y \cap W$ is infinite and $\mu(F_n) > \epsilon > 0$ holds for every $n \in Z$. Since $\mu$ is a finite measure it follows that

$$\mu\left(\cap_{n \in N} \bigcup \{F_k \mid k \in Z \text{ and } k > n\}\right) > \epsilon.$$
Thus we may choose $x$ and an infinite subset $V$ of $Z$ with $x \in F_k$ for all $k \in V$. Therefore $V \in [W]^\omega \cap \mathcal{R}$, which completes the proof.

11.1 Remark. In case $\mu$ is a regular Borel measure on $K$, a relatively simple case of Mathias’ theorem can be used in the proof of 11.10. In that case we may assume that the sets $F_n$ are closed (replacing $\epsilon$ by $\epsilon/2$ and each $F_n$ by a closed subset). Then the family $\mathcal{R}$ is actually a closed subset of $[N]^\omega$ as can be seen by a direct proof.

11.12 Theorem. Suppose $*\mathcal{M}$ is a minimal nonstandard model and $\mu$ is a Borel measure on a compact metric space $K$. If $E \subseteq *K$ is Loeb measurable then $\underline{\mu}(S(E)) < \bar{\mu}(\mu(E)) < \bar{\mu}(S(E))$

where $\underline{\mu}, \bar{\mu}$ are the inner and outer measures for $\mu$.

Proof. It suffices to prove $\mu(S(E)) < \bar{\mu}(\mu(E))$ since the other inequality follows from this one applied to $*K \setminus E$.

Let $B$ be a standard measurable set such that $B \subseteq S(E)$ and $\mu(B) = \mu(S(E))$. Let $A = *B \setminus E$. Notice $S(A) = \emptyset$ and $A$ is Loeb measurable so by Lemma 11.10.

$\bar{\mu}(\mu(A)) = 0$. Hence since $\bar{\mu}(\mu(A)) + \bar{\mu}(\mu(E)) > \bar{\mu}(\mu(B) = \underline{\mu}(S(E))$ we have $\bar{\mu}(\mu(E)) > \underline{\mu}(S(E))$ completing the proof.

Example II.6 shows that even if $E$ is internal and $*measurable, S(E)$ need not be $\mu$-measurable. The following Corollary gives a necessary and sufficient condition for $S(E)$ to be $\mu$-measurable when $E$ is an internal, $*Borel$ set in a minimal nonstandard model $*\mathcal{M}$.

11.13 Corollary. Let $*\mathcal{M}, \mu$ and $K$ be as in 11.10. For each internal $*Borel$ set $E \subseteq *K, S(E)$ is measurable with respect to the completion of $\mu$ if and only if there is a standard Borel set $B$ such that $E$ and $*B$ differ by a set of infinitesimal $*\mu$ measure.

Proof. If such a set $B$ exists, then $S(E)$ equals $B$ up to a $\mu$-null set, by II.9.

For the converse, suppose $S(E)$ is measurable with respect to the completion of $\mu$ and let $B \subseteq S(E)$ be a Borel set such that $S(E) \setminus B$ is a $\mu$ null set. Then $S(*B \setminus E)$ is empty and $S(E \setminus *B)$ is a $\mu$-null set, so that the symmetric difference of $*B$ and $E$ has infinitesimal $*\mu$ measure, by II.12.

11.14 Remark. Some restriction on the measure space of $\mu$ is necessary in order that II.10 should be true. For example, take Borel subsets $E_n$ of $(0, 1)^{(\omega)}$ as in Example II.6, so that the internal set $E$ determined by $(E_n)$ has internal measure $1/2$, yet $S(E)$ has inner measure 0 and outer measure 1. Then consider the measure space $\Omega = (0, 1)^{(\omega)} \setminus S(E)$ with the restricted measure $\mu$. Let $E_n' = E_n \setminus S(E)$, so $(E_n')$ are measurable subsets of $\Omega$. If $E'$ is the internal $*measurable$ subset of $*\Omega$ determined by the sequence $(E_n')$, then $S(E') = \emptyset$ yet $*\mu(E') = 1/2$.

11.15 Remark. Lemma II.10 which is the key step in the proof of Theorem II.12 has a very nice standard interpretation as follows. Suppose $D$ is a minimal ultrafilter on the set $\{1, 2, 3, \ldots \}$ and $E_1, E_2, \ldots$ is a sequence of Borel subsets of a compact metric space $K$ with $\mu$ a finite Borel measure on $K$. If $\inf \mu(E_n) > 0$ then there is a point $x \in K$ such that $\{n|x \in E_n\} \in D$. Notice this is a strengthening of the usual result that there is a point $x \in K$ which is in infinitely many $E_n$'s.
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