ON THE ITERATED LOGARITHM LAW FOR LOCAL TIME

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Abstract. If \( s(t, x) \) is the local time of a Brownian motion, we show that

\[
\theta(a) = \limsup_{t \to \infty} \inf_{|x| \leq \alpha^{1/2}(2 \log \log t)^{1/2}} s(t, x)(2t \log \log t)^{-1/2}
\]

satisfies

\[
((1 - a^{1/2}) \vee 0)^2 < \theta(a) < (2a)^{-1} \wedge 1.
\]

In particular, it follows from a result of Kesten that

\[
\limsup_{t \to \infty} s(t, x)(2t \log \log t)^{-1/2} = 1
\]

for all \( x \) a.s.

1. Introduction. Suppose \( B(t) \) is Brownian motion on a complete probability space \( (\Omega, \mathcal{F}, P) \) and \( s(t, x) = (d/dx) \int_0^t I(B(s) < x) \, ds \) (\( I(A) \) is the indicator function of \( A \)) is its jointly continuous local time. Since \( s(t, 0) \) is identical in law to \( \sup_{s < t} B(s) \), the law of the iterated logarithm implies that \( \limsup_{t \to \infty} s(t, x)(2t \log \log t)^{-1/2} = 1 \) a.s. for each real \( x \), where \( \phi(t) = (2t|\log|\log t||)^{1/2} \). In Kesten [1] it is shown that

\[
\limsup_{t \to \infty} \sup_{x \in \mathbb{R}} s(t, x)(2t \log \log t)^{-1/2} = 1 \quad \text{a.s. (1)}
\]

This implies that \( \limsup_{t \to \infty} s(t, x)\phi(t)^{-1} \leq 1 \) for all real \( x \) a.s. but leaves open the question as to whether or not there is equality for all \( x \) a.s. That there is equality for all \( x \) a.s. is an easy corollary of the following theorem.

Theorem 1. Let \( \Psi(t) = t^{1/2}(2|\log|\log t||)^{1/2} \). There is a nonincreasing function \( \theta(a) \) (\( a > 0 \)) such that

(a) \( \limsup_{t \to \infty} \inf_{|x| \leq \Psi(t)} s(t, x)\phi(t)^{-1} = \theta(a) \) a.s. for all \( a > 0 \),

(b) \( \theta(a) < (2a)^{-1} \wedge 1 \) for all \( a > 0 \),

(c) \( \theta(a) > (1 - a^{1/2})^2 \) for all \( a < 1 \). \( \square \)

The method of proof is that in Kesten [1] but some simplification occurs due to the use of a maximal inequality for submartingales.

2. Main result.

Notation 2. If \( a > 0 \), let \( T(a) = \inf\{t > 0|s(t, 0) > a\} \).

Lemma 3. If \( a > 0 \), \( s(T(a), x) \) is a martingale in \( x > 0 \) and satisfies

\[
E(e^{-\lambda s(T(a), x)}) = \exp\{-\lambda \alpha(1 + 2|x|)^{-1}\} \quad (\lambda > 0).
\]

Proof. By Knight [2], \( s(T(a), x) \) is a diffusion in \( x > 0 \) with infinitesimal generator \( 2yd^2/dy^2 \), and in particular is a nonnegative local martingale. Moreover,
(2) is derived in the proof of Corollary 1.2 in Knight [2]. It follows from (2) that $E(s(T(a), x)) = a < \infty$ for all $x$ and hence $s(T(a), x)$ is a supermartingale by Fatou’s lemma. Since $E(s(T(a), x))$ is independent of $x$, we see that $s(T(a), x)$ must in fact be a martingale.

**Proof of Theorem 1.** Since $\lim \sup \sup_{t \to \infty} \inf_{|x| < \alpha \Psi(t)} s(t, x) \phi(t)^{-1}$ is measurable with respect to the tail $\sigma$-field of a Brownian motion, a well-known zero-one law implies that the above expression is a.s. equal to a nonnegative constant $\theta(\alpha)$. Clearly $\theta(\alpha) < 1$ because $\lim \sup_{t \to \infty} s(t, 0) \phi(t)^{-1} = 1$ a.s. Moreover if $t > \epsilon$, then

$$\inf_{|x| < \alpha \Psi(t)} s(t, x) \phi(t)^{-1} \leq \int_{-\alpha \Psi(t)}^{\alpha \Psi(t)} s(t, x) \phi(t)^{-1} \, dx (2\alpha \Psi(t))^{-1}$$

$$< t (2\alpha \phi(t) \Psi(t))^{-1} = (\alpha)^{-1}.$$

It remains to show (c). Fix $\alpha \in [0, 1)$ and $\alpha_1 \in (0, 1)$. Then choose $\alpha_2 \in (\alpha_1 \vee \alpha, 1)$ such that $\alpha_1 \in (\alpha_{1/2}^2 - \alpha_{1/2}^2)$ or, equivalently, $\alpha < (\alpha_{1/2}^2 - \alpha_{1/2}^2)^2$. The usual proof of the law of the iterated logarithm allows us to choose $t > 1$ such that $P(T_k < t^k$ infinitely often) = 1, where $T_k = T(\alpha_2 \phi(t^k))$. Therefore

$$P\left( \inf_{|x| < \alpha \Psi(t_k)} s(T_k, x) \phi(t_k)^{-1} > \alpha_1 \text{ infinitely often} \right)$$

$$> P\left( \inf_{|x| < \alpha \Psi(t_k)} s(T_k, x) > \alpha_1 \phi(t_k) \text{ and } T_k < t^k \text{ infinitely often} \right)$$

$$> P\left( \sup_{|x| < \alpha \Psi(t_k)} s(T_k, 0) - s(T_k, x) > (\alpha_2 - \alpha_1) \phi(t_k) \text{ only finitely often} \right). \quad (3)$$

Use a maximal inequality for submartingales and Lemma 3 to see that if $\lambda > 0$, then

$$P\left( \sup_{|x| < \alpha \Psi(t^k)} s(T_k, 0) - s(T_k, x) > (\alpha_2 - \alpha_1) \phi(t^k) \right)$$

$$< 2 P\left( \sup_{0 < x < \alpha \Psi(t^k)} s(T_k, 0) - s(T_k, x) > (\alpha_2 - \alpha_1) \phi(t^k) \right)$$

$$< 2 \exp\{-\lambda(\alpha_2 - \alpha_1) \phi(t^k)\} E\left( \exp\{\lambda(s(T_k, 0) - s(T_k, \alpha \Psi(t^k)))\} \right)$$

$$= 2 \exp\{-\lambda(\alpha_2 - \alpha_1) \phi(t^k) + \lambda \alpha_2 \phi(t^k) - \lambda \alpha_2 \phi(t^k)(1 + 2\lambda \phi(t^k))^{-1} \}$$

(by (2))

$$= 2 \exp\{-\lambda(\alpha_2(1 + 2\lambda \Psi(t^k))^{-1} - \alpha_1)\}. \quad (4)$$

An elementary calculus argument shows that (4) has a minimum value of

$$2 \exp\{- (\alpha_{1/2}^2 - \alpha_{1/2}^2)^2 \alpha^{-1}\log\log t^k \} \quad (5)$$

when $\lambda = ((\alpha_2 \alpha_1^{-1})^{1/2} - 1)(2\alpha \Psi(t^k))^{-1}$. Since $(\alpha_{1/2}^2 - \alpha_{1/2}^2)^2 > \alpha$, (5) is summable over $k$ and therefore

$$P\left( \sup_{|x| < \alpha \Psi(t^k)} s(T_k, 0) - s(T_k, x) > (\alpha_2 - \alpha_1) \phi(t^k) \text{ only finitely often} \right) = 1.$$
by the Borel-Cantelli lemma. It follows from (3) that
\[ \limsup_{k \to \infty} \inf_{|x| < \alpha \psi(T_k)} s(T_k, x) \phi(T_k)^{-1} \geq \alpha_1 \text{ a.s.} \]
for all \( \alpha_1 < (1 - \alpha^{1/2})^2 \), and hence \( \theta(\alpha) > (1 - \alpha^{1/2})^2 \). □

Since \( \theta(\infty) = 0 \), if \( h(t) \) satisfies \( \lim_{t \to \infty} h(t) \Psi(t)^{-1} = +\infty \) then
\[ \limsup_{t \to \infty} \inf_{|x| < h(t)} s(t, x) \phi(t)^{-1} = 0 \text{ a.s.,} \]
and since \( \theta(0^+) = 1 \), if \( h(t) \) satisfies \( \lim_{t \to \infty} h(t) \psi(t)^{-1} = 0 \) then
\[ \limsup_{t \to \infty} \inf_{|x| < h(t)} s(t, x) \phi(t)^{-1} = 1 \text{ a.s.} \]

This latter result (with \( \lim_{t \to \infty} h(t) = \infty \)), coupled with (1), gives us the following corollary.

**Corollary 4.** For \( \omega \) outside a single null set, \( \limsup_{t \to \infty} s(t, x) \phi(t)^{-1} = 1 \) for all \( x \). □

**Remark 5.** A trivial modification of the proof of Theorem 1 shows that for all \( \alpha > 0 \) there is a constant \( \theta^1(\alpha) \) satisfying (b) and (c) of Theorem 1 and also
\[ \limsup_{t \to 0^+} \inf_{|x| < \alpha^{1/2}(2 \log \log r^{-1})^{-1/2}} s(t, x)(2t \log \log r^{-1})^{-1/2} = \theta^1(\alpha) \text{ a.s.} \] □

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**References**


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