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GROWTH OF POLYNOMIALS WITH ZEROS OUTSIDE A CIRCLE

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ABSTRACT. Let $P(z)$ be a polynomial of degree n having all its zeros in $|z| > k > 1$. For $k = 1$, it is known that

$$\operatorname*{Max}_{|z|=R>1} |P(z)| < \frac{R^n + 1}{2} \operatorname*{Max}_{|z|=1} |P(z)|.$$

In this paper we consider the case $k > 1$ and obtain a sharp result.

Let $P(z)$ be a polynomial of degree n such that $\operatorname{Max}_{|z|=1}|P(z)| = 1$, then

$$\operatorname*{Max}_{|z|=R>1} |P(z)| \leq R^n. \tag{1}$$

Inequality (1) is a simple deduction from the Maximum Modulus Principle (see [4, p. 346] or [3, Vol. I, p. 137, problem III 269]). It was conjectured by Erdös and first proved by Lax [2] that, if $P(z) \neq 0$ in $|z| < 1$ then

$$\operatorname*{Max}_{|z|=1} |P'(z)| \leq n/2. \tag{2}$$

Ankeny and Rivlin [1] used (2) to prove the following theorem.

THEOREM A. *If $P(z)$ is a polynomial of degree n with* $\operatorname{Max}_{|z|=1}|P(z)| = 1$ *and $P(z)$ has no zeros in the disk $|z| < 1$, then*

$$\operatorname*{Max}_{|z|=R>1} |P(z)| \leq \frac{R^n + 1}{2}. \tag{3}$$

The result is best possible and equality in (3) *holds for $P(z) = (z^n + 1)/2$.*

We shall generalize Theorem A by proving the following theorems.

THEOREM 1. *If $P(z)$ is a polynomial of degree n with* $\operatorname{Max}_{|z|=1}|P(z)| = 1$ *and $P(z)$ has no zeros in the disk $|z| < k$ where $K \geq 1$, then*

$$\operatorname*{Max}_{|z|=R>1} |P(z)| \leq \frac{(R^n + 1)(R + k)^n}{(R + k)^n + (1 + Rk)^n}. \tag{4}$$

Theorem 1 is a generalization of Theorem A in a compact form but unfortunately with the exception of $n = 1$, (4) does not appear to be sharp for $k > 1$.

However, a precise estimate is given by the following theorem.

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THEOREM 2. *If $P(z)$ is a polynomial of degree n with* $\operatorname{Max}_{|z|=1}|P(z)| = 1$ *and $P(z)$ has no zeros in the disk $|z| < k$ where $k \geqslant 1$, then for $R \geqslant 1$ we have*

$$\operatorname*{Max}_{|z|=R}|P(z)| \leqslant \frac{R^n + k^n}{1 + k^n} \quad \text{for } R \geqslant k^2 \tag{5}$$

and

$$\operatorname*{Max}_{|z|=R}|P(z)| \leqslant \frac{(R + k)^n}{(1 + k)^n} \quad \text{for } 1 \leqslant R \leqslant k^2. \tag{6}$$

The result is best possible with equality in (5) *for $P(z) = (z^n + k^n)/(1 + k^n)$ and in* (6) *for $P(z) = (z + k)^n/(1 + k)^n$.*

For the proofs of these theorems we need the following Lemma.

LEMMA. *If $P(z)$ is a polynomial of degree n with* $\operatorname{Max}_{|z|=1}|P(z)| = 1$, *then*

$$|P(Re^{i\theta})| + |Q(Re^{i\theta})| \leqslant R^n + 1, \qquad 0 \leqslant \theta < 2\pi, \tag{7}$$

where $Q(z) = z^n\overline{P(1/\bar{z})}$ and $R \geqslant 1$.

PROOF OF THE LEMMA. Since $|P(z)| \leqslant 1$ for $|z| = 1$, therefore, if α is a complex number such that $|\alpha| > 1$, it follows from Rouche's theorem that the polynomial $F(z) = P(z) - \alpha$ does not vanish inside the unit circle. Thus the polynomial

$$G(z) = z^n\overline{F(1/\bar{z})} = z^n\overline{P(1/\bar{z})} - \bar{\alpha}z^n = Q(z) - \bar{\alpha}z^n$$

has all its zeros in $|z| \leqslant 1$ and $|F(z)| = |G(z)|$ for $|z| = 1$. Hence $|G(z)/F(z)|$ is analytic on and inside the unit circle, and on the boundary $|G(z)/F(z)| = 1$. By the Maximum Modulus Principle it follows that $|G(z)| \leqslant |F(z)|$ for $|z| \leqslant 1$. Replacing z by $1/\bar{z}$ we get $|z^n\overline{G(1/\bar{z})}| \leqslant |z^n\overline{F(1/\bar{z})}|$ for $|z| \geqslant 1$. Since $z^n\overline{G(1/\bar{z})} \equiv F(z)$, we conclude that $|F(z)| \leqslant |G(z)|$ for $|z| \geqslant 1$. Therefore, in particular

$$|F(Re^{i\theta})| \leqslant |G(Re^{i\theta})| \quad \text{where } R \geqslant 1 \text{ and } 0 \leqslant \theta < 2\pi.$$

This gives

$$|P(Re^{i\theta}) - \alpha| \leqslant |Q(Re^{i\theta}) - \bar{\alpha}R^ne^{in\theta}|, \qquad 0 \leqslant \theta < 2\pi.$$

Now choosing an argument of α such that $|Q(Re^{i\theta}) - \bar{\alpha}R^ne^{in\theta}| = |\alpha|R^n - |Q(Re^{i\theta})|$, we obtain

$$|P(Re^{i\theta})| - |\alpha| \leqslant |\alpha|R^n - |Q(Re^{i\theta})|, \qquad 0 \leqslant \theta < 2\pi.$$

Equivalently

$$|P(Re^{i\theta})| + |Q(Re^{i\theta})| \leqslant |\alpha|(R^n + 1), \qquad 0 \leqslant \theta < 2\pi.$$

Letting now $|\alpha| \to 1$, we obtain (7) and this proves the Lemma.

PROOF OF THEOREM 1. Since the polynomial $P(z)$ has all its zeros in $|z| \geqslant k \geqslant 1$, we write

$$P(z) = \prod_{j=1}^{n}\left(z - r_je^{i\theta_j}\right) \quad \text{where } r_j \geqslant k, j = 1, 2, \ldots, n.$$

Then $Q(z) = z^n\overline{P(1/\bar{z})} = \prod_{j=1}^{n}(1 - zr_je^{-i\theta_j})$ so that clearly we have for $0 \leq \theta < 2\pi$ and for $R > 1$

$$\begin{aligned}|P(Re^{i\theta})/Q(Re^{i\theta})| &= \prod_{j=1}^{n}\left|\left(Re^{i\theta} - r_je^{i\theta_j}\right)/\left(1 - Rr_je^{i(\theta-\theta_j)}\right)\right| \\ &= \prod_{j=1}^{n}\left|\left(Re^{i(\theta-\theta_j)} - r_j\right)/\left(1 - r_jRe^{i(\theta-\theta_j)}\right)\right| \\ &\leq \prod_{j=1}^{n}(R + r_j)/(1 + Rr_j) \\ &\leq \prod_{j=1}^{n}(R + k)/(1 + Rk) = (R + k)^n/(1 + Rk)^n.\end{aligned}$$

This implies

$$((1 + Rk)^n/(R + k)^n)|P(Re^{i\theta})| \leq |Q(Re^{i\theta})|, \qquad 0 \leq \theta < 2\pi.$$

Combining this with the conclusion of the Lemma we obtain

$$(1 + (1 + Rk)^n/(R + k)^n)|P(Re^{i\theta})| \leq R^n + 1, \qquad 0 \leq \theta < 2\pi.$$

Consequently

$$\operatorname*{Max}_{|z|=R>1}|P(z)| \leq \frac{(R^n + 1)(R + k)^n}{(R + k)^n + (1 + Rk)^n},$$

which is the desired result.

Proof of Theorem 2. Since all the zeros of $P(z)$ lie in $|z| \geq k \geq 1$, we write as before

$$P(z) = \prod_{j=1}^{n}\left(z - r_je^{i\theta_j}\right) \quad \text{where } r_j \geq k, j = 1, 2, \ldots, n.$$

Then $Q(z) = z^n\overline{P(1/\bar{z})} = \prod_{j=1}^{n}(1 - zr_je^{-i\theta_j})$. Now

$$\begin{aligned}|P(k^2e^{i\theta})/Q(e^{i\theta})| &= \prod_{j=1}^{n}\left|\left(k^2e^{i\theta} - r_je^{i\theta_j}\right)/\left(1 - r_je^{i(\theta-\theta_j)}\right)\right| \\ &= \prod_{j=1}^{n}\left|\left(k^2e^{i(\theta-\theta_j)} - r_j\right)/\left(1 - r_je^{i(\theta-\theta_j)}\right)\right| \\ &\leq \prod_{j=1}^{n}k = k^n \quad \text{for } 0 \leq \theta < 2\pi.\end{aligned}$$

Therefore, we have

$$|P(k^2z)| \leq k^n|Q(z)| \quad \text{for } |z| = 1. \tag{8}$$

Since all the zeros of $Q(z)$ lie in $|z| \leq 1/k \leq 1$, it follows from the Maximum Modulus Principle that

$$|P(k^2z)| \leq k^n|Q(z)| \quad \text{for } |z| \geq 1.$$

Hence in particular we have

$$|P(k^2Re^{i\theta})| \leq k^n|Q(Re^{i\theta})| \quad \text{for } R \geq 1 \text{ and } 0 \leq \theta < 2\pi.$$

This inequality with the help of the Lemma gives

$$|P(k^2Re^{i\theta})| + k^n|P(Re^{i\theta})| \leq k^n(R^n + 1), \qquad 0 \leq \theta < 2\pi. \tag{9}$$

Now for every given θ, $0 \leq \theta < 2\pi$, and for $R \geq 1$, we have either

$$|P(k^2Re^{i\theta})| - |P(Re^{i\theta})| \leq (k^n - 1)R^n \tag{10}$$

or

$$|P(k^2Re^{i\theta})| - |P(Re^{i\theta})| > (k^n - 1)R^n. \tag{11}$$

Inequality (10) yields with the help of inequality (9)

$$(1 + k^n)|P(k^2Re^{i\theta})| \leq (k^{2n}R^n + k^n)$$

and inequality (11) yields with the help of inequality (9)

$$\begin{aligned}(1 + k^n)|P(Re^{i\theta})| &< |P(k^2Re^{i\theta})| + k^n|P(Re^{i\theta})| - (k^n - 1)R^n \\ &\leq k^n(R^n + 1) - (k^n - 1)R^n = R^n + k^n.\end{aligned}$$

Thus we have either

$$\operatorname*{Max}_{|z|=R} |P(z)| \leq (R^n + k^n)/(1 + k^n) \quad \text{for } R \geq k^2 \tag{12}$$

or

$$\operatorname*{Max}_{|z|=R} |P(z)| < (R^n + k^n)/(1 + k^n) \quad \text{for } R \geq 1. \tag{13}$$

From (12) and (13), it follows that, we have in any case

$$\operatorname*{Max}_{|z|=R} |P(z)| \leq \frac{R^n + k^n}{1 + k^n} \quad \text{for } R \geq k^2$$

and this proves the inequality (5).

To prove the inequality (6), we observe that for $1 \leq R \leq k^2$ and $0 \leq \theta < 2\pi$

$$\begin{aligned}|P(Re^{i\theta})/P(e^{i\theta})| &= \prod_{j=1}^{n} \left|\left(Re^{i\theta} - r_je^{i\theta_j}\right)/\left(e^{i\theta} - r_je^{i\theta_j}\right)\right| \\ &= \prod_{j=1}^{n} \left|\left(Re^{i(\theta-\theta_j)} - r_j\right)/\left(e^{i(\theta-\theta_j)} - r_j\right)\right| \\ &\leq \prod_{j=1}^{n} (R + r_j)/(1 + r_j) \\ &\leq \prod_{j=1}^{n} (R + k)/(1 + k) = (R + k)^n/(1 + k)^n.\end{aligned}$$

Consequently

$$\operatorname*{Max}_{|z|=R} |P(z)| \leq \frac{(R + k)^n}{(1 + k)^n} \quad \text{for } 1 \leq R \leq k^2.$$

This is the inequality (6) and the theorem is completely established.

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