A COUNTEREXAMPLE FOR COMMUTATION IN TENSOR PRODUCTS OF C*-ALGEBRAS

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Abstract. An example is given to show the failure of the analogue for C*-algebras of the commutation theorem for von Neumann tensor products.

Let $A$, $B$, $C$, $D$ be C*-algebras with $A \subseteq C$ and $B \subseteq D$. Tomiyama [10, p. 29] has raised the question as to whether $(A \otimes B)^c = A^c \otimes B^c$. In this context $\otimes$ denotes the spatial tensor product and $(A \otimes B)^c$ (respectively $A^c$, $B^c$) is the relative commutant of $A \otimes B$ (respectively $A$, $B$) in $C \otimes D$ (respectively $C$, $D$). It is easy to see that the inclusion $A^c \otimes B^c \subseteq (A \otimes B)^c$ is always valid. In the special case where $C$ has an identity $1$, $A = C1$ and $B = D$, the question has an affirmative answer [7, Theorem 1], and the result has been generalized to the case of an arbitrary C*-tensor norm [1], [4]. It is therefore tempting to conjecture that the question has an affirmative answer at least in the case where $A = C1$ (but $B$ is an arbitrary C*-subalgebra of $D$). In this note we present a counterexample based on results of Choi [5], Wassermann [13] and Voiculescu [11].

We begin by recalling some facts about slice maps [9]. Let $A$ and $B$ be C*-algebras and let $\phi \in A^*$. The right slice map $R_\phi : A \otimes B \to B$ is the unique bounded linear mapping with the property that $R_\phi(a \otimes b) = \phi(a)b$ ($a \in A$, $b \in B$). A triple $(A, B, J)$, where $J$ is a closed two-sided ideal of $B$, is said to verify the slice map conjecture [12] if whenever $x \in A \otimes B$ and $R_\phi(x) \in J$ for all $\phi \in A^*$ then $x \in A \otimes J$. It is well known that $(A, B, J)$ verifies the slice map conjecture if and only if $A \otimes J$ is the kernel of the canonical *-homomorphism $\mu : A \otimes B \to A \otimes (B/J)$. This is because

$$\ker \mu = \{ x \in A \otimes B | R_\phi(x) \in J \text{ for all } \phi \in A^* \}.$$

The following result is implicit in [13, 2.5]. Although we shall apply it in a rather special case, we state it in the given form since it may be of independent interest.

Proposition. Suppose that $B/J$ is a nuclear C*-algebra. Then the triple $(A, B, J)$ verifies the slice map conjecture.

Proof. Suppose that $(A, B, J)$ does not verify the slice map conjecture. The canonical *-isomorphism of the algebraic tensor product $A \odot (B/J)$ into $(A \otimes B)/(A \otimes J)$ induces on $A \odot (B/J)$ a C*-norm which is distinct from the least C*-norm since $\ker \mu \neq A \otimes J$. This contradicts the nuclearity of $B/J$. 

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Remark. Since any quotient of a nuclear C*-algebra is nuclear [6], it follows from the Proposition that \((A, B, J)\) verifies the slice map conjecture whenever \(B\) is a nuclear C*-algebra. It is shown in [2] that in fact it suffices to assume that \(B\) is just a C*-subalgebra of some nuclear C*-algebra in order to conclude that \((A, B, J)\) verifies the slice map conjecture.

We now give the counterexample. Let \(C = C^*(F_2)\), the full C*-algebra of the free group on two generators, and let \(J\) be the kernel of the canonical *-homomorphism from \(C\) onto the C*-algebra of the left regular representation of \(F_2\). Since \(C\) is separable it follows from [8, 3.7.5] that there is a faithful (nondegenerate) representation \(\pi\) of \(C\) on a separable Hilbert space \(H\) (of infinite dimension). By [5, Corollary 2] \(\pi(C)\) contains no nonzero compact operator. It follows that we may regard \(C\) as a C*-subalgebra of the Calkin algebra (\(D\) say) associated with \(H\). Since \(\pi\) was nondegenerate we may assume that the identity \(1\) of \(D\) lies in \(C\). Let \(A = Cl\) and let \(B = (J + Cl)^c\) (the relative commutant of \(J + Cl\) in \(D\)). Then, with the relative commutants taken as indicated in the opening paragraph, we have the following result.

**Theorem.** \((A \otimes B)^c \neq A^c \otimes B^c\).

**Proof.** Since \(J + Cl\) is a separable C*-subalgebra of \(D\) it follows from [11] (see also [3, p. 345]) that \((J + Cl)^{cc} = J + Cl\). Thus \(B^c = J + Cl\) and so \(A^c \otimes B^c = C \otimes (J + Cl)\). By [13, 2.7, Remark] there exists \(x \in C \otimes C\) \((\subseteq C \otimes D)\) such that \(x \notin C \otimes J\) and \(R_\phi(x) \in J\) for all \(\phi \in C^*\) (where \(R_\phi\) is the right slice map \(C \otimes C \to C\) associated with \(\phi\)). Applying the Proposition to the triple \((C, J + Cl, J)\) we see that \(x \notin C \otimes (J + Cl)\).

Let \(b \in B\). For \(\phi \in C^*\) we have

\[
R_\phi\left[ x(1 \otimes b) - (1 \otimes b)x \right] = R_\phi(x)b - bR_\phi(x) = 0.
\]

Hence \(x(1 \otimes b) - (1 \otimes b)x = 0\) [9, Theorem 1], and so \(x \in (A \otimes B)^c\).

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**References**


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