PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 4, April 1981

SPHERICAL SUMMABILITY OF DIFFERENTIATED MULTIPLE FOURIER SERIES[1]

M. J. KOHN

ABSTRACT. We prove a theorem on the Bochner-Riesz summability of formally differentiated multiple Fourier series.

1. Suppose $f(t)$ is a periodic function of a single variable. Let $S[f]$ be its Fourier series and let $S^{(r)}[f]$ be the series obtained by formally differentiating $S[f]$ r times. The following result, see [**6**, vol. II, p. 60], is well known: *If $f(t_0)$ has an rth symmetric derivative equal to s then $S^{(r)}[f]$ is summable (C, α) at t_0 to sum s provided $\alpha > r$.*

In this paper we establish a p-dimensional analogue to the preceding result. For functions of several variables we use a definition of "weighted symmetric derivative" based upon taking weighted spherical averages and expanding them into power series. The weights we use are surface harmonics.

In p dimensions let $x = (x_1, \ldots, x_p)$, $n = (n_1, \ldots, n_p)$, $x' = |x|^{-1}x$, $T^p = \{x \in E^p | -\pi < x_i \leqslant \pi,\ i = 1, \ldots, p\}$, and $\Sigma = \{x \in E^p | |x| = 1\}$. Let $\Omega(\eta)$ be a surface harmonic of order ν; that is, the restriction to Σ of a polynomial that is harmonic and homogeneous of degree ν. Let r be an integer of the form $r = \nu + 2k$, where k is a nonnegative integer. Suppose $F(x)$ is defined in a neighborhood of $x_0 \in E^p$. We will say *$F(x_0)$ has an rth Ω-derivative equal to s* if F is integrable over each sphere $|x - x_0| = t$, for t small, and if

$$(2\pi)^{-p/2}\int_{\eta\in\Sigma} F(x_0 + t\eta)\Omega(\eta)ds(\eta) = a_\nu t^\nu + a_{\nu+2}t^{\nu+2} + \cdots + a_r t^r + o(t^r) \tag{1.1}$$

as $r \to 0$, where

$$a_r = \frac{2^{-p/2-r-1}s}{((r-\nu)/2)!\Gamma((r+\nu+p)/2)}$$

and $ds(\eta)$ represents the element of $(p-1)$-dimensional surface area.

The previous definition is taken from [**3**] where the following result is proved: *Suppose $P(x)$ is a harmonic polynomial homogeneous of degree ν and let $\Omega(\eta)$ be the restriction of $P(x)$ to Σ. Let $r = \nu + 2k$ where k is a nonnegative integer. If $F(x)$ and*

Received by the editors March 10, 1980.

1980 *Mathematics Subject Classification*. Primary 42B05.

Key words and phrases. Multiple Fourier series, Bochner-Riesz summability, weighted symmetric derivative.

[1]This work has been partially supported by the National Science Foundation under grant #MCS-8003584.

all partial derivatives of F of order $\leqslant r + 1$ are continuous in a neighborhood of $x_0 \in E^p$, then $F(x_0)$ has an rth Ω-derivative with value $P(\text{grad})\Delta^k F(x_0)$.

Now let $F(x)$ be defined for $x \in T^p$. Let $S[F]$ denote the multiple Fourier series of F. If $P(x)$ is a polynomial for $x \in E^p$, let $P(D)S[F]$ denote the series obtained by formally applying the differential operator $P(D)$ to $S[F]$.

We will say a multiple trigonometric series $\Sigma c_n \exp(in \cdot x)$ is Bochner-Riesz-β summable at x_0 to s if

$$\lim_{R\to\infty} \sum_{n\in\mathbb{Z}^p} c_n \exp(in \cdot x)\varphi(|n|/R) = s \tag{1.2}$$

where

$$\varphi(u) = \begin{cases} (1 - u^2)^\beta & \text{if } 0 \leqslant u < 1, \\ 0 & \text{if } u \geqslant 1. \end{cases}$$

THEOREM. *Let $P(x)$ be a harmonic polynomial homogeneous of degree ν and let $\Omega(\eta)$ be the restriction of $P(x)$ to the unit sphere. Let r be a number of the form $r = \nu + 2k$, where k is a nonnegative integer. Suppose $F(x_0)$ has an rth Ω-derivative equal to s. Then $P(D)\Delta^k S[F]$ is Bochner-Riesz-β summable at x_0 to s, provided $\beta > r + (p - 1)/2$.*

This theorem has been proven when $\nu = r = 0$ in [1] and when $\nu = 0$, $r > 0$ in [4]. The proof here is based on a modification of the arguments from [1] and [4].

2. Proof of the Theorem. We are given

$$(2\pi)^{-p/2}\int_{\eta\in\Sigma} F(x_0 + t\eta)\Omega(\eta)ds(\eta)$$
$$= a_\nu t^\nu + a_{\nu+2}t^{\nu+2} + \cdots + a_r t^r + o(t^r)$$

as $t \to 0$. We may assume without loss of generality, see [4], that $a_\nu = a_{\nu+2} = \cdots = a_r = 0$. Hence

$$(2\pi)^{-p/2}\int_{\eta\in\Sigma} F(x_0 + t\eta)\Omega(\eta)ds(\eta) = o(t^r) \tag{2.1}$$

as $t \to 0$. Let

$$S[F] = \sum_{n\in\mathbb{Z}^p} c_n e^{in\cdot x}.$$

Then

$$P(D)\Delta^k S[F] = \sum_{n\in\mathbb{Z}^p} (-1)^k |n|^{2k} P(in) c_n e^{in\cdot x}.$$

Let

$$\sigma_R(x) = \sum_{|n|<R} (-1)^k |n|^{2k} P(in) c_n e^{in\cdot x}(1 - |n|^2/R^2)^\beta$$
$$= \sum_{n\in\mathbb{Z}^p} (-1)^k |n|^{2k} P(in) c_n e^{in\cdot x}\varphi(|n|/R). \tag{2.2}$$

We must show $\sigma_R(x) \to 0$ as $R \to \infty$.

We begin by establishing the following relation:

$$\sigma_R(x) = (-1)^k R^{r+1} \int_0^\infty F_\Omega(x, t) H_\varphi(Rt)\, dt \tag{2.3}$$

where

$$F_\Omega(x, t) = (2\pi)^{-p/2} \int_{\eta \in \Sigma} F(x + t\eta)\Omega(\eta) ds(\eta) \tag{2.4}$$

and

$$H_\varphi(c) = c^{p/2} \int_0^\infty u^{r+p/2} \varphi(u) J_{\nu+(p-2)/2}(cu)\, du. \tag{2.5}$$

We first prove (2.3) for the special case when $F(x)$ is an exponential monomial, $F(x) = \exp(in \cdot x)$. In this case,

$$\sigma_R(x) = (-1)^k |n|^{2k} P(in) \exp(in \cdot x) \varphi(|n|/R) \tag{2.6}$$

and

$$\begin{aligned} F_\Omega(x, t) &= (2\pi)^{-p/2} \int_{\eta \in \Sigma} e^{in\cdot(x+t\eta)} \Omega(\eta) ds(\eta) \\ &= e^{in\cdot x} (2\pi)^{-p/2} \int_{\eta \in \Sigma} e^{in\cdot t\eta} \Omega(\eta) ds(\eta). \end{aligned} \tag{2.7}$$

The last integral on the right of (2.7) can be evaluated by the Funk-Hecke theorem, see [**2**, p. 181]: *if $\xi \in \Sigma$ and $f(h)$ is continuous for $-1 < h < 1$ then*

$$\begin{aligned} &\int_{\eta \in \Sigma} f(\xi \cdot \eta) \Omega(\eta)\, ds(\eta) \\ &\quad = \Omega(\xi) \cdot \frac{2^{-\nu+1} \pi^{(p-1)/2}}{\Gamma(\nu + (p-1)/2)} \int_{-1}^1 f^{(\nu)}(h)(1 - h^2)^{\nu + (p-3)/2}\, dh. \end{aligned} \tag{2.8}$$

We apply (2.8) with $\xi = n' = |n|^{-1} n$ and $f(h) = \exp(i|n|th)$. Then

$$\begin{aligned} \int_{\eta \in \Sigma} \exp(in \cdot t\eta) \Omega(\eta) ds(\eta) &= \int_{\eta \in \Sigma} f(n' \cdot \eta) \Omega(\eta) ds(\eta) \\ &= \Omega(n') \frac{2^{-\nu+1} \pi^{(p-1)/2} (i|n|t)^\nu}{\Gamma(\nu + (p-1)/2)} \\ &\quad \cdot \int_{-1}^1 e^{i|n|th} (1 - h^2)^{\nu + (p-3)/2}\, dh. \end{aligned} \tag{2.9}$$

The last integral on the right can be evaluated by formula 4 of [**5**, p. 48]. We get

$$\begin{aligned} &\int_{-1}^1 e^{i|n|th} (1 - h^2)^{\nu + (p-3)/2}\, dh \\ &\quad = \frac{\Gamma(\nu + (p-1)/2)}{\pi^{-1/2} (1/2|n|t)^{\nu + (p-2)/2}} J_{\nu + (p-2)/2}(|n|t). \end{aligned} \tag{2.10}$$

Combining (2.7), (2.9), and (2.10), we obtain

$$F_\Omega(x, t) = e^{in\cdot x} \Omega(n') i^\nu (|n|t)^{-l} J_{\nu+l}(|n|t), \tag{2.11}$$

where $l = (p - 2)/2$.

We next use *Hankel's repeated integral*, see [**5**, p. 456]: *if $g(u)$ is continuous at s and if $\int_0^\infty g(u) u^{1/2}\, du < \infty$* then

$$g(s) = \int_0^\infty cJ_{\nu+l}(cs)\,dc \int_0^\infty g(u)J_{\nu+l}(cu)u\,du.$$

We use this formula with $g(s) = \varphi(s)s^{r+l}$. Then

$$\varphi(s)s^{r+l} = \int_0^\infty cJ_{\nu+l}(cs)\,dc \int_0^\infty \varphi(u)u^{r+l}J_{\nu+l}(cs)u\,du.$$

Hence

$$\varphi(s) = s^{-r-l}\int_0^\infty c^{-l}J_{\nu+l}(cs)H_\varphi(c)\,dc$$

where $H_\varphi(c)$ is defined by (2.5). Therefore

$$\varphi(R^{-1}|n|) = (R^{-1}|n|)^{-r-l}\int_0^\infty c^{-l}J_{\nu+l}(cR^{-1}|n|)H_\varphi(c)\,dc.$$

We change variables in the last integral, letting $c = Rt$. We get

$$\begin{aligned}\varphi(R^{-1}|n|) &= (R^{-1}|n|)^{-r-l}\int_0^\infty (Rt)^{-l}J_{\nu+l}(|n|t)H_\varphi(Rt)R\,dt \\ &= R^{r+1}|n|^{-r}\int_0^\infty (|n|t)^{-l}J_{\nu+l}(|n|t)H_\varphi(Rt)\,dt. \qquad (2.12)\end{aligned}$$

Returning to (2.6),

$$\begin{aligned}\sigma_R(x) &= (-1)^k|n|^{2k}P(in)\exp(in\cdot x)\varphi(R^{-1}|n|) \\ &= (-1)^k i^\nu|n|^{\nu+2k}\Omega(n')\exp(in\cdot x)\varphi(R^{-1}|n|) \\ &= (-1)^k R^{r+1}\int_0^\infty F_\Omega(x,t)H_\varphi(Rt)\,dt,\end{aligned}$$

by (2.11). This establishes (2.3) for the special case when $F(x)$ is an exponential monomial. In the general case, formula (2.3) can be proved for arbitrary $F(x)$ using an approximation process. The argument is given in detail in the proof of Theorem V of [1] and is omitted here.

We next simplify $H_\varphi(c)$ by first changing variables and then using formula (1) from [5, p. 373].

$$\begin{aligned}H_\varphi(c) &= c^{l+1}\int_0^\infty u^{r+l+1}\varphi(u)J_{\nu+l}(uc)\,du \\ &= c^{l+1}\int_0^1 u^{r+l+1}(1-u^2)^\beta J_{\nu+l}(uc)\,du \\ &= c^{l+1}\int_0^{\pi/2}\cos^{2\beta+1}\theta\,\sin^{r+l+1}\theta J_{\nu+l}(uc)\,du \\ &= c^{l+1}\int_0^{\pi/2}\cos^{2\beta+1}\theta(1-\cos^2\theta)^k\sin^{\nu+l+1}\theta J_{\nu+l}(c\sin\theta)\,d\theta \\ &= \sum_{j=0}^k(-1)^j\binom{k}{j}c^{l+1}\int_0^{\pi/2}\cos^{2j+2\beta+1}\theta\,\sin^{\nu+l+1}\theta J_{\nu+l}(c\sin\theta)\,d\theta \\ &= \sum_{j=0}^k(-1)^j\binom{k}{j}c^{l+1}2^{j+\beta}\Gamma(j+\beta+1)c^{l-\beta-j}J_{\nu+l+\beta+j+1}(c) \\ &= \sum_{j=0}^k A_jc^{l-\beta-j}J_{\nu+l+\beta+j+1}(c). \qquad (2.13)\end{aligned}$$

We now complete the proof of the Theorem. By (2.3) and (2.13),

$$\sigma_R(x) = (-1)^k R^{r+1} \int_0^\infty F_\Omega(x, t) H_\varphi(Rt)\, dt$$
$$= \sum_{j=0}^{k} (-1)^k A_k R^{r+1} \int_0^\infty F_\Omega(x, t)(Rt)^{l-\beta-j} J_{\nu+l+\beta+j+1}(Rt)\, dt.$$

We will show each term of the last sum tends to 0.

Fix j and fix $\delta > 0$. Let $R > \delta^{-1}$. Then

$$R^{r+1} \int_0^\infty F_\Omega(x, t)(Rt)^{l-\beta-j} J_{\nu+l+j+1}(Rt)\, dt$$
$$= R^{r+1} \int_0^{1/R} + R^{r+1} \int_{1/R}^{\delta} + R^{r+1} \int_\delta^\infty = \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

To estimate I we use the inequality $|J_s(z)| \leqslant A|z|^s$, for $|z| < 1$.

$$I = R^{r+1} \int_0^{1/R} o(t^r)(Rt)^{l-\beta-j} O(Rt)^{\nu+l+\beta+j+1}\, dt$$
$$= R^{r+2l+\nu+2} \int_0^{1/R} o(t^{r+2l+\nu+1})\, dt = o(1),$$

as $R \to \infty$.

To estimate II we use the inequality $|J_s(z)| \leqslant A|z|^{-1/2}$ for $|z| \geqslant 1$.

$$\mathrm{II} = R^{r+1} \int_{1/R}^{\delta} o(t^r)(Rt)^{l-\beta-j} O(Rt)^{-1/2}\, dt$$
$$= R^{r+l-\beta-j+1/2} \int_{1/R}^{\delta} o(t^{r+l-\beta-j-1/2})\, dt = o(1),$$

as $R \to \infty$.

For the estimate of III we recall that $F(x)$ is periodic, so if $\lambda > 0$ then

$$\int_{\delta+\lambda}^{\delta+\lambda+1} |F_\Omega(x, t)|\, dt \leqslant C.$$

$$|\mathrm{III}| \leqslant R^{r+1} \sum_{\lambda=0}^{\infty} \int_{\delta+\lambda}^{\delta+\lambda+1} |F_\Omega(x, t)|(Rt)^{l-\beta-j} O(Rt)^{-1/2}\, dt$$
$$\leqslant R^{r+1} \sum_{\lambda=0}^{\infty} C \cdot (R(\delta + \lambda))^{l-\beta-j-1/2}$$
$$= R^{l+r-\beta-j+1/2} \sum_{\lambda=0}^{\infty} C(\delta + \lambda)^{l-\beta-j-1/2} = o(1),$$

as $R \to \infty$. Note we needed $\beta > r + (p - 1)/2$ for the estimates of II and III. This completes the proof of the theorem.

REFERENCES

1. S. Bochner, *Summation of multiple Fourier series by spherical means*, Trans. Amer. Math. Soc. **40** (1936), 175–207.

2. H. Hochstadt, *The functions of mathematical physics*, Interscience, New York, 1971.

3. M. Kohn, *On differentiation of multiple trigonometric series*, Illinois J. Math. (to appear).

4. V. Shapiro, *Circular summability C of double trigonometric series*, Trans. Amer. Math. Soc. **16** (1954), 223–233.

5. G. N. Watson, *Theory of Bessel functions*, 2nd ed., Cambridge Univ. Press, Cambridge, 1966.

6. A. Zygmund, *Trigonometric series*, vols. I and II, Cambridge Univ. Press, Cambridge, 1968.

DEPARTMENT OF MATHEMATICS, BROOKLYN COLLEGE OF THE CITY UNIVERSITY OF NEW YORK, BROOKLYN, NEW YORK 11210