SPHERICAL SUMMABILITY
OF DIFFERENTIATED MULTIPLE FOURIER SERIES

M. J. KOHN

Abstract. We prove a theorem on the Bochner-Riesz summability of formally differentiated multiple Fourier series.

1. Suppose \( f(t) \) is a periodic function of a single variable. Let \( S[f] \) be its Fourier series and let \( S^{(r)}[f] \) be the series obtained by formally differentiating \( S[f] \) \( r \) times. The following result, see [6, vol. II, p. 60], is well known: If \( f(t_0) \) has an \( r \)th symmetric derivative equal to \( s \) then \( S^{(r)}[f] \) is summable \( (C, \alpha) \) at \( t_0 \) to sum \( s \) provided \( \alpha > r \).

In this paper we establish a \( p \)-dimensional analogue to the preceding result. For functions of several variables we use a definition of “weighted symmetric derivative” based upon taking weighted spherical averages and expanding them into power series. The weights we use are surface harmonics.

In \( p \) dimensions let \( x = (x_1, \ldots, x_p) \), \( n = (n_1, \ldots, n_p) \), \( x' = |x|^{-1}x \), \( T^p = \{ x \in E^p \mid -\pi < x_i < \pi, \ i = 1, \ldots, p \} \), and \( \Sigma = \{ x \in E^p \mid |x| = 1 \} \). Let \( \Omega(\eta) \) be a surface harmonic of order \( \nu \); that is, the restriction to \( \Sigma \) of a polynomial that is harmonic and homogeneous of degree \( \nu \). Let \( r \) be an integer of the form \( r = \nu + 2k \), where \( k \) is a nonnegative integer. Suppose \( F(x) \) is defined in a neighborhood of \( x_0 \in E^p \). We will say \( F(x_0) \) has an \( r \)th \( \Omega \)-derivative equal to \( s \) if \( F \) is integrable over each sphere \( |x - x_0| = t \), for \( t \) small, and if

\[
(2\pi)^{-p/2} \int_{\eta \in \Sigma} F(x_0 + t\eta) \Omega(\eta) d\sigma(\eta) = a_r t^r + a_{r+2} t^{r+2} + \cdots + a_r t^r + o(t^r)
\]

as \( r \to 0 \), where

\[
a_r = \frac{2^{-p/2-r-1/2}}{((r-\nu)/2)!\Gamma((r+\nu+p)/2)}
\]

and \( d\sigma(\eta) \) represents the element of \((p - 1)\)-dimensional surface area.

The previous definition is taken from [3] where the following result is proved: Suppose \( P(x) \) is a harmonic polynomial homogeneous of degree \( \nu \) and let \( \Omega(\eta) \) be the restriction of \( P(x) \) to \( \Sigma \). Let \( r = \nu + 2k \) where \( k \) is a nonnegative integer. If \( F(x) \) and

Received by the editors March 10, 1980.

1980 Mathematics Subject Classification. Primary 42B05.

Key words and phrases. Multiple Fourier series, Bochner-Riesz summability, weighted symmetric derivative.

1This work has been partially supported by the National Science Foundation under grant #MCS-8003584.

© 1981 American Mathematical Society

0002-9939/81/0000-0168/$02.50
all partial derivatives of $F$ of order $< r + 1$ are continuous in a neighborhood of $x_0 \in E^p$, then $F(x_0)$ has an $r$th $\Omega$-derivative with value $P(\text{grad})\Delta^k F(x_0)$.

Now let $F(x)$ be defined for $x \in T^p$. Let $S[F]$ denote the multiple Fourier series of $F$. If $P(x)$ is a polynomial for $x \in E^p$, let $P(D)S[F]$ denote the series obtained by formally applying the differential operator $P(D)$ to $S[F]$.

We will say a multiple trigonometric series $\sum c_n \exp(in \cdot x)$ is Bochner-Riesz-$\beta$ summable at $x_0$ to $s$ if

\[
\lim_{R \to \infty} \sum_{n \in \mathbb{Z}^p} c_n \exp(in \cdot x) \varphi(|n|/R) = s
\]

where

\[
\varphi(u) = \begin{cases} 
(1 - u^2)^\beta & \text{if } 0 < u < 1, \\
0 & \text{if } u > 1.
\end{cases}
\]

**Theorem.** Let $P(x)$ be a harmonic polynomial homogeneous of degree $\nu$ and let $\Omega(\eta)$ be the restriction of $P(x)$ to the unit sphere. Let $r$ be a number of the form $r = \nu + 2k$, where $k$ is a nonnegative integer. Suppose $F(x_0)$ has an $r$th $\Omega$-derivative equal to $s$. Then $P(D)\Delta^k S[F]$ is Bochner-Riesz-$\beta$ summable at $x_0$ to $s$, provided $\beta > r + (p - 1)/2$.

This theorem has been proven when $\nu = r = 0$ in [1] and when $\nu = 0$, $r > 0$ in [4]. The proof here is based on a modification of the arguments from [1] and [4].

2. Proof of the Theorem. We are given

\[
(2\pi)^{-p/2} \int_{\eta \in \Sigma} F(x_0 + t\eta)\Omega(\eta) d\eta = a_t r^r + a_{r+2} r^{r+2} + \cdots + a_r r^r + o(t^r)
\]

as $t \to 0$. We may assume without loss of generality, see [4], that $a_r = a_{r+2} = \cdots = a_r = 0$. Hence

\[
(2\pi)^{-p/2} \int_{\eta \in \Sigma} F(x_0 + t\eta)\Omega(\eta) d\eta = o(t^r)
\]

as $t \to 0$. Let

\[
S[F] = \sum_{n \in \mathbb{Z}^p} c_n e^{in \cdot x}.
\]

Then

\[
P(D)\Delta^k S[F] = \sum_{n \in \mathbb{Z}^p} (-1)^k |n|^{2k} P(in) c_n e^{in \cdot x}.
\]

Let

\[
\sigma_R(x) = \sum_{|n| < R} (-1)^k |n|^{2k} P(in) c_n e^{in \cdot x} (1 - |n|^2 / R^2)^\beta
\]

\[
\sigma_R(x) = \sum_{n \in \mathbb{Z}^p} (-1)^k |n|^{2k} P(in) c_n e^{in \cdot x} \varphi(|n|/R).
\]

We must show $\sigma_R(x) \to 0$ as $R \to \infty$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
We begin by establishing the following relation:

\[
\sigma_R(x) = (-1)^k R^{k+1} \int_0^\infty F_u(x, t) H_\varphi(Rt) \, dt
\]  

(2.3)

where

\[
F_u(x, t) = (2\pi)^{-p/2} \int_{\eta \in \Sigma} F(x + t\eta) \Omega(\eta) \, d\eta
\]  

(2.4)

and

\[
H_\varphi(c) = c^{p/2} \int_0^\infty u^{r+p/2} \varphi(u) J_{r+(p-2)/2}(cu) \, du.
\]  

(2.5)

We first prove (2.3) for the special case when \( F(x) \) is an exponential monomial, \( F(x) = \exp(in \cdot x) \). In this case,

\[
\sigma_R(x) = (-1)^k |n|^{2k} P^k(\exp(in \cdot x)) \varphi(|n|/R)
\]  

(2.6)

and

\[
F_u(x, t) = (2\pi)^{-p/2} \int_{\eta \in \Sigma} e^{in \cdot (x + t\eta)} \Omega(\eta) \, d\eta
\]  

(2.7)

The last integral on the right of (2.7) can be evaluated by the Funk-Hecke theorem, see [2, p. 181]: if \( \xi \in \Sigma \) and \( f(h) \) is continuous for \(-1 < h < 1\) then

\[
\int_{\eta \in \Sigma} f(\xi \cdot \eta) \Omega(\eta) \, d\eta = \Omega(\xi) \cdot \frac{2^{-r+1} \Gamma(r-1/2)}{\Gamma(v + (p - 1)/2)} \int_{-1}^1 f(\eta)(1 - h^2)^{r+(p-3)/2} \, dh.
\]  

(2.8)

We apply (2.8) with \( \xi = n' = |n|^{-1} n \) and \( f(h) = \exp(i|n|h) \). Then

\[
\int_{\eta \in \Sigma} \exp(in \cdot t\eta) \Omega(\eta) \, d\eta = \int_{\eta \in \Sigma} f(n' \cdot \eta) \Omega(\eta) \, d\eta
\]  

\[
= \Omega(n') \cdot \frac{2^{-r+1} \Gamma(r-1/2)}{\Gamma(v + (p - 1)/2)} \int_{-1}^1 f(\eta)(1 - h^2)^{r+(p-3)/2} \, dh.
\]  

(2.9)

The last integral on the right can be evaluated by formula 4 of [5, p. 48]. We get

\[
\int_{-1}^1 e^{i|n|h}(1 - h^2)^{r+(p-3)/2} \, dh = \frac{\Gamma(v + (p - 1)/2)}{\pi^{-1/2}(1/2||n||^2)^{r+(p-3)/2}} J_{r+(p-2)/2}(|n||t|).
\]  

(2.10)

Combining (2.7), (2.9), and (2.10), we obtain

\[
F_u(x, t) = e^{in \cdot x} \Omega(n') i^{r(|n||t|)} J_{r+(p-2)/2}(|n||t|),
\]  

(2.11)

where \( l = (p - 2)/2 \).

We next use Hankel’s repeated integral, see [5, p. 456]: if \( g(u) \) is continuous at \( s \) and if \( \int_0^\infty g(u) u^{(p-1)/2} \, du < \infty \) then

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
\[ g(s) = \int_0^\infty cJ_{r+1}(cs) \, dc \int_0^\infty g(u)J_{r+1}(cu) \, du. \]

We use this formula with \( g(s) = \varphi(s)s^{r+l} \). Then
\[
\varphi(s)s^{r+l} = \int_0^\infty cJ_{r+1}(cs) \, dc \int_0^\infty \varphi(u)u^{r+l}J_{r+1}(cs) \, du.
\]

Hence
\[
\varphi(s) = s^{-r-l} \int_0^\infty c^{-l}J_{r+1}(cs) H_\varphi(c) \, dc
\]
where \( H_\varphi(c) \) is defined by (2.5). Therefore
\[
\varphi(R^{-1}|n|) = (R^{-1}|n|)^{-r-l} \int_0^\infty c^{-l}J_{r+1}(cR^{-1}|n|) H_\varphi(c) \, dc.
\]

We change variables in the last integral, letting \( c = Rt \). We get
\[
\varphi(R^{-1}|n|) = (R^{-1}|n|)^{-r-l} \int_0^\infty (Rt)^{-l}J_{r+1}(|n|t) H_\varphi(Rt) R \, dt
\]
\[= R^{r+1}|n|^{-r} \int_0^\infty (|n|t)^{-l}J_{r+1}(|n|t) H_\varphi(Rt) \, dt. \quad (2.12)\]

Returning to (2.6),
\[
\sigma_R(x) = (-1)^k |n|^{2k} P(n) \exp(in \cdot x) \varphi(R^{-1}|n|)
\]
\[= (-1)^k i^n |n|^{*+2k} \Omega(n') \exp(in \cdot x) \varphi(R^{-1}|n|)
\]
\[= (-1)^k R^{r+1} \int_0^\infty F_\varphi(x, t) H_\varphi(Rt) \, dt,
\]
by (2.11). This establishes (2.3) for the special case when \( F(x) \) is an exponential monomial. In the general case, formula (2.3) can be proved for arbitrary \( F(x) \) using an approximation process. The argument is given in detail in the proof of Theorem V of [1] and is omitted here.

We next simplify \( H_\varphi(c) \) by first changing variables and then using formula (1) from [5, p. 373].
\[
H_\varphi(c) = c^{l+1} \int_0^\infty u^{r+l+1} \varphi(u)J_{r+1}(uc) \, du
\]
\[= c^{l+1} \int_0^1 u^{r+l+1}(1 - u^2) \beta J_{r+1}(uc) \, du
\]
\[= c^{l+1} \int_0^{\pi/2} \cos^{2\beta+1} \theta \sin^{r+l+1} \theta J_{r+1}(uc) \, d\theta
\]
\[= c^{l+1} \int_0^{\pi/2} \cos^{2\beta+1} \theta (1 - \cos^2 \theta)^k \sin^{r+l+1} \theta J_{r+1}(c \sin \theta) \, d\theta
\]
\[= \sum_{j=0}^k (-1)^j \binom{k}{j} c^{l+1} \int_0^{\pi/2} \cos^{2\beta+1} \theta \sin^{r+l+1} \theta J_{r+1}(c \sin \theta) \, d\theta
\]
\[= \sum_{j=0}^k (-1)^j \binom{k}{j} c^{l+1} \Gamma(j + \beta + 1) c^{l-\beta} J_{r+l+\beta+j+1}(c)
\]
\[= \sum_{j=0}^k A_j c^{l-\beta} J_{r+l+\beta+j+1}(c). \quad (2.13)\]
We now complete the proof of the Theorem. By (2.3) and (2.13),

$$\sigma_R(x) = (-1)^k R^{r+1} \int_0^\infty F_\Omega(x, t) \psi(Rt) \, dt$$

$$= \sum_{j=0}^k (-1)^k A_k R^{r+1} \int_0^\infty F_\Omega(x, t)(Rt)^{-\beta-j} J_{r+j+\beta+j+1}(Rt) \, dt.$$

We will show each term of the last sum tends to 0.

Fix $j$ and fix $\delta > 0$. Let $R > \delta^{-1}$. Then

$$R^{r+1} \int_0^\infty F_\Omega(x, t)(Rt)^{-\beta-j} J_{r+j+\beta+j+1}(Rt) \, dt$$

$$= R^{r+1} \int_0^{1/R} + R^{r+1} \int_{1/R}^{\delta} + R^{r+1} \int_{\delta}^\infty = I + II + III.$$

To estimate I we use the inequality $|J_\delta(z)| < A|z|^\delta$, for $|z| < 1$.

$$I = R^{r+1} \int_0^{1/R} o(t')(Rt)^{-\beta-j} O(Rt)^{s+\lambda+j+1} \, dt$$

$$= R^{r+2l+s+2} \int_0^{1/R} o(t^{r+2l+s+1}) \, dt = o(1),$$

as $R \to \infty$.

To estimate II we use the inequality $|J_\delta(z)| < A|z|^{-1/2}$ for $|z| > 1$.

$$II = R^{r+1} \int_{1/R}^{\delta} o(t')(Rt)^{-\beta-j} O(Rt)^{-1/2} \, dt$$

$$= R^{r+1-\beta-j+1/2} \int_{1/R}^{\delta} o(t^{r+1-\beta-j-1/2}) \, dt = o(1),$$

as $R \to \infty$.

For the estimate of III we recall that $F(x)$ is periodic, so if $\lambda > 0$ then

$$\int_{\delta+\lambda}^{\delta+\lambda+1} |F_\Omega(x, t)| \, dt < C.$$

$$|III| < R^{r+1} \sum_{\lambda=0}^\infty \int_{\delta+\lambda}^{\delta+\lambda+1} |F_\Omega(x, t)|(Rt)^{-\beta-j} O(Rt)^{-1/2} \, dt$$

$$< R^{r+1} \sum_{\lambda=0}^\infty C \cdot (R(\delta + \lambda))^{1-\beta-j-1/2}$$

$$= R^{l+r-\beta-j+1/2} \sum_{\lambda=0}^\infty C(\delta + \lambda)^{l-\beta-j-1/2} = o(1),$$

as $R \to \infty$. Note we needed $\beta > r + (p - 1)/2$ for the estimates of II and III. This completes the proof of the theorem.
REFERENCES


Department of Mathematics, Brooklyn College of the City University of New York, Brooklyn, New York 11210