ALMOST COMPACTNESS AND DECOMPOSABILITY
OF INTEGRAL OPERATORS

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Abstract. Let \((X, \mu), (Y, \nu)\) be finite measure spaces and \(1 < q < \infty, 1 < p < q\). An integral operator \(\text{Int}(k) : L^q(\nu) \to L^p(\mu)\) becomes compact, if we cut away a suitably chosen subset of \(X\) of arbitrarily small measure. As a consequence we prove that \(\text{Int}(k)\) may be written as the sum of a Carleman operator and an orderbounded integral operator, where the orderbounded part may be chosen to be compact and of arbitrarily small norm.

1. Introduction. \((X, \mathcal{X}, \mu)\) and \((Y, \mathcal{Y}, \nu)\) will denote finite measure spaces. For \(1 < p, q < \infty\) we call an operator \(T : L^q(\nu) \to L^p(\mu)\) integral, if there is a measurable kernel-function \(k(x, y)\) on \(X \times Y\) such that for \(g \in L^q(\nu)\)

\[ Tg(x) = \int_Y k(x, y)g(y) \, d\nu(y) \quad \mu\text{-a.e.} \]

The integrand is required to be Lebesgue-integrable for \(\mu\text{-a.e.} \ x \in X\) (cf. [7] or [9]). In this case we write \(T = \text{Int}(k)\).

There are two well-behaved subclasses of integral operators: \(\text{Int}(k)\) is called Carleman if, for \(\mu\text{-a.e.} \ x \in X, k(x, \cdot) \in L^r(\nu)\) where \(r^{-1} + q^{-1} = 1\). The operator \(\text{Int}(k)\) is called orderbounded if it transforms orderbounded sets into orderbounded sets or equivalently if \(|k|\) also defines an integral operator from \(L^q(\nu)\) to \(L^p(\mu)\). In this case we call \(\text{Int}(|k|)\) the modulus or absolute value of \(\text{Int}(k)\).

Let us specify the following notation. If \(g \in L^\infty(\mu)\) we denote by \(P_g\) the multiplication operator \(f \mapsto f \cdot g\) on \(L^p(\mu)\). If \(g = \chi_A\) is a characteristic function we write \(P_A\) for \(P_{\chi_A}\).

2. Preliminaries. In this section we recall known results for later reference.

2.1. Theorem (Nikišin, [11, Theorem 4]). Let \(0 < q < \infty\) and \(T : L^q(\nu) \to L^p(\mu)\) be a positive, continuous operator. For \(\epsilon > 0\) there is an \(A \subseteq X, \mu(X \setminus A) < \epsilon\) and such that \(P_A \circ T\) takes its values in \(L^q(\mu)\).

2.2. Theorem (Maurey, [10, Proposition 9]). Let \(0 < p < q < \infty\) and \(T : L^q(\nu) \to L^p(\mu)\) be a positive, continuous operator. For \(r^{-1} = p^{-1} - q^{-1}\) there is a strictly positive function \(g \in L^\infty(\mu)\) such that \(g^{-1} \in L^r(\mu)\) and \(P_g \circ T\) takes its values in \(L^q(\mu)\).

We also need a technical result, which follows easily from [9, Theorems 4.7 and 5.12].

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2.3. Lemma. Let $1 < q < \infty$, $1 < p < \infty$ and $k(x, y) > 0$ be such that $\text{Int}(k)$ defines an operator from $L^q(\nu)$ to $L^p(\mu)$. Let $k_n(x, y) > 0$ be such that $k = \sum_{n=1}^{\infty} k_n$. 

(a) $\sum_{n=1}^{\infty} \text{Int}(k_n)$ converges unconditionally to $\text{Int}(k)$ in the strong operator topology of $B(L^q(\nu), L^p(\mu))$.

(b) If $1 < q < \infty$ and $\text{Int}(k)$ is compact then the above sum converges unconditionally in the norm of $B(L^q(\nu), L^p(\mu))$.

3. Almost compactness of positive integral operators.

3.1. Theorem. Let $1 < q < \infty$ and $k(x, y) > 0$ be such that $\text{Int}(k)$ defines an operator from $L^q(\nu)$ to $L^q(\mu)$. Given $r < \infty$ we may find $g \in L^{\infty}(\mu)$ such that $g^{-1} \in L'(\mu)$ and $P_g \circ \text{Int}(k) : L^q(\nu) \to L^q(\mu)$ is compact.

Proof. Let us start with the easy case $q = \infty$. It is an old result, dating back to Dunford’s paper [4] in 1936, that a $\sigma$*-continuous $T : L^\infty(\nu) \to L^\infty(\mu)$ is integral iff for $\epsilon > 0$ there is $A \subseteq X$, $\mu(X \setminus A) < \epsilon$ and such that $P_A \circ T$ is compact (see also [5] and [12]). So find a partition $(A_n)_{n=1}^{\infty}$ of $X$ such that $P_{A_n} \circ \text{Int}(k)$ is compact and, given $r < \infty$, find a nullsequence $(\alpha_n)_{n=1}^{\infty}$ of strictly positive scalars such that $g^{-1} = \sum_{n=1}^{\infty} \alpha_n \chi_{A_n} \in L'(\mu)$. It is easy to check that $P_g \circ \text{Int}(k)$ is compact.

Now assume that $1 < q < \infty$. Given $r < \infty$ find $1 < p < q$ such that $r^{-1} - q^{-1}$. The operator $\text{Int}(k)$ is a compact operator from $L^q(\nu)$ to $L^p(\nu)$ (cf. [1] or [9, Theorem 5.4]; compare also [3]). Let $k_n = k \cdot \chi_{(n-1 < k < n)}$ and deduce from 2.3(b) that $\sum_{n=1}^{\infty} \text{Int}(k_n)$ converges to $\text{Int}(k)$ unconditionally in the norm of $B(L^q(\nu), L^p(\nu))$. So we may find a sequence $0 = n_0 < n_1 < \cdots < n_m < \ldots$ such that for $m > 2$ 

$$\left\| \sum_{n=n_{m-1}}^{n_m-1} \text{Int}(k_n) \right\|_{B(L^q, L^p)} < 2^{-m}.$$ 

Let $\tilde{k}_m = m \sum_{n=n_{m-1}}^{n_m-1} k_n$, and $\tilde{k} = \sum_{n=1}^{\infty} \tilde{k}_m$. Clearly $\tilde{k} > k$ but $\text{Int}(\tilde{k}) = \sum_{m=1}^{\infty} \text{Int}(\tilde{k}_m)$ is still a continuous (even compact, but we shall not need this) operator from $L^q(\nu)$ to $L^p(\mu)$. We may apply Maurey’s factorization theorem (2.2 above) to find $g \in L^{\infty}(\mu)$ such that $g^{-1} \in L'(\mu)$ and such that $P_g \circ \text{Int}(\tilde{k})$ takes its values in $L^q(\nu)$. From 2.3(a) 

$$P_g \circ \text{Int}(\tilde{k}) = \sum_{m=1}^{\infty} m \left( \sum_{n=n_{m-1}}^{n_m-1} \text{Int}(g \cdot k_n) \right),$$

the sum converging unconditionally in the strong operator topology of $B(L^q(\nu), L^q(\mu))$. This implies that the sum 

$$P_g \circ \text{Int}(k) = \sum_{m=1}^{\infty} \left( \sum_{n=n_{m-1}}^{n_m-1} \text{Int}(g \cdot k_n) \right)$$

converges in the norm of $B(L^q(\nu), L^q(\mu))$. As each of the summands is clearly compact the operator $P_g \circ \text{Int}(k) : L^q(\nu) \to L^q(\mu)$ is compact.

3.2. Remark. The theorem does not hold for $q = 1$. Let $T : L^1(\nu) \to L^1[0, 1]$ be a positive surjective operator, where $(Y, \nu)$ is a purely atomic measure space (i.e.
$L^1(\nu)$ is isometric to $l^1$. Then $T$ is integral but for every positive $g \in L^\infty(\mu)$, which does not vanish identically, the operator $P_g \circ \text{Int}(k)$ is not compact.

However, we have the following result by duality.

3.3. Corollary. Let $1 < p < \infty$ and $k(y, x) > 0$ such that $\text{Int}(k)$ defines an operator from $L^p(\mu)$ to $L^p(\nu)$. Given $r < \infty$ we may find $g \in L^\infty(\mu)$ such that $g^{-1} \in L^r(\mu)$ and $\text{Int}(k) \circ P_g : L^p(\mu) \to L^p(\nu)$ is compact.

4. Almost compactness of general integral operators.

4.1. Theorem. Let $1 < q < \infty$ and $1 < p < q$ and let $\text{Int}(k) : L^q(\nu) \to L^p(\mu)$ be an integral operator. For $\epsilon > 0$ there is $A \subseteq X$ with $\mu(X \setminus A) < \epsilon$ such that both $P_A \circ \text{Int}(k)$ and its modulus $P_A \circ \text{Int}(|k|)$ are compact operators from $L^q(\nu)$ to $L^q(\mu)$.

Proof. Write $k = k_1 - k_2 + ik_3 - ik_4$, where $k_j > 0$. Each $\text{Int}(k_j)$ defines a positive continuous operator from $L^q(\nu)$ to $L^q(\mu)$. By Nikisin's theorem (2.1 above) we may find $B_j \subseteq X$, $\mu(X \setminus B_j) < \epsilon/8$ such that $P_{B_j} \circ \text{Int}(k_j)$ is a positive continuous operator from $L^q(\nu)$ to $L^q(\mu)$. It is an easy consequence of Theorem 3.1 that we may find $A_j \subseteq B_j$, $\mu(X \setminus A_j) < \epsilon/4$, such that $P_{A_j} \circ \text{Int}(k_j)$ is compact from $L^q(\nu)$ to $L^q(\mu)$. For $A = \cap_{j=1}^4 A_j$, the operator $P_A \circ \text{Int}(k)$ satisfies the requirements.

4.2. Remark. In the situation of Theorem 4.1, it is not possible to find a big set $B$ on the left-hand side (i.e. from $Y$) so that $\text{Int}(k) \circ P_B$ is compact. For example let $k$ be the kernel on $[0, 1] \times [0, 1]$, $k(x, y) = 2^{n/2} \cdot r_n(y)$ if $x \in [2^{-n}, 2^{-(n-1)}]$, where $r_n$ denotes the $n$th Rademacher function. Then $\text{Int}(k) : L^2[0, 1] \to L^2[0, 1]$ is such an example.

Theorem 4.1 is a strengthening of the known result of “twosided cutting off”, which seems to be due to Korotkov [8].

4.3. Remark. What happens in the case $p > q$? If $\text{Int}(k) : L^q(\nu) \to L^p(\mu)$ is given, then for $q \geq 1$ the above theorem applies and provides a compact operator $P_A \circ \text{Int}(k)$ from $L^q(\nu)$ to $L^q(\mu)$. One would like to have the operator compact from $L^q(\nu)$ to $L^p(\mu)$ but this is only possible for few pairs of indices as is shown in the following proposition.

4.4. Proposition. (a) Let $1 < q < \infty$ and $p = \infty$; for every continuous operator $T : L^q(\nu) \to L^\infty(\mu)$ and $\epsilon > 0$ there is an $A \subseteq X$ with $\mu(X \setminus A) < \epsilon$ such that $P_A \circ T : L^q(\nu) \to L^\infty(\mu)$ is compact.

(b) On the other hand, for $1 < q < p < \infty$ and for $q = 1, p = \infty$ there are integral operators $\text{Int}(k) : L^q(\nu) \to L^p(\mu)$ such that for every $A \subseteq X$, $\mu(A) > 0$ the operator $P_A \circ T : L^q(\nu) \to L^p(\mu)$ is not compact.

Proof. (a) This result was known to A. Grothendieck [6]. Let us phrase it in the terminology of [13]: $L^q(\nu)$ is Asplund for $1 < q < \infty$ hence $T$ (ball($L^q(\nu)$)) is equimeasurable, which is just what we have to prove.

(b) For $q = 1$ and $1 < p < \infty$ let $T$ be a positive surjective operator from $L^1$ (represented as $L^1(\nu)$ over a finite measure space $(Y, \nu)$) onto $L^p[0, 1]$ (resp. onto the subspace $C[0, 1]$ of $L^\infty[0, 1]$, if $p = \infty$).
If \( 1 < q < p < \infty \) then there are operators of potential type from \( L^q[0, 1] \) to \( L^p[0, 1] \) that are not compact (cf. [8, p. 147 ff.]). It is clear that an operator of potential type may not be made compact by restricting to a subset of positive measure.

5. Decomposition of integral operators.

5.1. Theorem. Let \( 1 < q < \infty, 1 < p < q \) and \( \text{Int}(k): L^q(\nu) \to L^p(\mu) \) an integral operator. Given \( \varepsilon > 0 \) we may write \( k = k^c + k^0 \) where \( \text{Int}(k^c) \) is a Carleman operator from \( L^q(\nu) \) to \( L^p(\mu) \) and \( \text{Int}(k^0) \) as well as its modulus \( \text{Int}(|k^0|) \) are compact operators from \( L^q(\nu) \) to \( L^q(\mu) \) of norm less than \( \varepsilon \).

Proof. We start with the trivial case \( q = \infty \) and \( 1 < p < \infty \). Every \( \text{Int}(k): L^\infty(\nu) \to L^p(\mu) \) is automatically Carleman, hence we may choose \( k^c = k \) and \( k^0 = 0 \).

Let now \( 1 < q < \infty, 1 < p < q \). By Theorem 4.1 we may find a partition \((A_i)_{i=1}^\infty\) of \( X \) such that for \( k_i(x,y) = \chi_{A_i}(x) \cdot k(x,y) \) the operator \( \text{Int}(k_i) \) is compact from \( L^q(\nu) \) to \( L^q(\mu) \). By Lemma 2.3 we may find numbers \( n_i \) such that

\[
\|\text{Int}(|k_i|) - \text{Int}(|k_i| \cdot \chi_{|k_i| < n_i})\| < \varepsilon/2^i.
\]

Let \( k_i^c = k_i \cdot \chi_{|k_i| < n_i} \) and \( k_i^0 = k_i - k_i^c \) and define

\[
k^c = \sum_{i=1}^\infty k_i^c \quad \text{and} \quad k^0 = \sum_{i=1}^\infty k_i^0.
\]

It is now easy to verify the asserted properties of \( k^c \) and \( k^0 \).

5.2. Remark. We do not know whether for arbitrary \( 1 < p, q < \infty \) an integral operator \( \text{Int}(k): L^q(\nu) \to L^p(\mu) \) may be decomposed into a Carleman and an orderbounded part. We know that this is possible in some cases not covered by 5.1.
For \( p = q = 1 \), for example, this is trivially possible as every continuous operator \( T: L^1(\nu) \to L^1(\mu) \) is orderbounded. However, we do not have the full strength of 5.1 in this case. The operator from Remark 3.2 may not be decomposed in such a way as to make the orderbounded part compact or arbitrarily small in norm.

References


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