SHORTER NOTES

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elegant and polished character, for which there is no other outlet.

A REGULAR SPACE WHICH IS NOT COMPLETELY REGULAR

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Abstract. We give a simple example of a regular space which is not completely
regular.

Every completely regular space is a regular space. The converse is false, however
the construction of known counterexamples are quite complicated (compare [4] and
[2, Example 2.4.21]). The aim of this note is to provide an elementary construction
of a regular noncompletely regular space.

The underlying set of our space $X$ is the closed upper half-plane $y > 0$ plus an
additional point $a$. All points $(x, y)$ with $y > 0$ are assumed to be isolated. The
basic neighborhoods of $(x, 0)$ contain $(x, 0)$ and all but finitely many points from
the union of two segments $I_x = \{(x, y): 0 < y < 2\}$ and $I'_x = \{(x + y, y): 0 < y
< 2\}$. And the basic neighborhoods of the point $a$ have the form $U_n(a) = \{a\} \cup
\{(x, y): x > n\}$ where $n = 1, 2, \ldots$

It is easy to check that $X$ is a regular space–neighborhoods of points from the
half-plane are closed and open sets and $\text{cl}_X U_{n+2}(a) \subseteq U_n(a)$ for every natural
number $n$.

We shall prove that $X$ is not completely regular. The set $A \equiv \{(x, y): x < 1\}$ is
closed in $X$. Let $f$ be an arbitrary continuous real-valued function on $X$ such that
$f(A) = \{0\}$. We show that $f(a) = 0$. Since the set $f^{-1}(0)$ is closed in $X$ it suffices
to prove that for every natural number $n$ the set $K_n = f^{-1}(0) \cap \{(x, 0): n - 1 < x <
n\}$ is infinite. We proceed by induction. Obviously, the set $K_1 = \{(x, 0): 0 < x <
1\}$ is infinite. Assume now that there is a countably infinite subset $C$ of $K_n$. For
each $(c, 0) \in C$ the set $I'_c - f^{-1}(0)$ is an $F_\sigma$ set not containing $(c, 0)$ and therefore
it is countable. Hence the projection $P$ of the union $\bigcup \{I'_c - f^{-1}(0): (c, 0) \in C\}$
onto the line $y = 0$ is countable. Let $F = \{(x, 0): n < x < n + 1\} - P$. Clearly $F$
is infinite. For every $(x, 0) \in F$ the segment $I_x$ meets each of the sets $I'_c \cap f^{-1}(0)$
with $(c, 0) \in C$, so that by the closedness of $f^{-1}(0)$ we have $(x, 0) \in f^{-1}(0)$. It
follows that $F \subset f^{-1}(0)$. Thus the set $K_{n+1}$ is infinite and the proof is concluded.

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**Remark.** If we adjoin to the space \( X \) a new point \( b \) with the basic neighborhoods of the form \( U_n(b) = \{ b \} \cup \{(x,y): x < -n \text{ and } x + n + 2 < y\}, \) \( n = 1, 2, \ldots \), then the obtained space \( Y \) is regular and \( f(a) = f(b) \) for every continuous real-valued function \( f \) on \( Y \). Using the space \( Y \) and the ideas from [1] or [3] one can obtain a relatively simple example of a regular space on which all continuous real-valued functions are constant.

**References**

1. E. K. van Douwen, *A regular space on which every continuous real-valued function is constant*, Nieuw Arch. Wisk. 20 (1972), 143–145.

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