CONVEXITY OF THE DOMINANT EIGENVALUE OF
AN ESSENTIALLY NONNEGATIVE MATRIX

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ABSTRACT. The dominant eigenvalue of a real $n \times n$ matrix $A$ with nonnegative elements off the main diagonal is a convex function of the diagonal of $A$. We give a short proof using Trotter's product formula and a theorem on log-convexity due to Kingman.

An $n \times n$ real matrix $A$ with nonnegative elements $a_{ij}$ ($i \neq j$) off the main diagonal is called essentially nonnegative. Such an $A$ has an eigenvalue $r(A)$, called the dominant eigenvalue, that is real and greater than or equal to the real part of any other eigenvalue of $A$. $r(A)$ need not be the eigenvalue of $A$ with the largest modulus, but when every element of $A$ is nonnegative, $r(A)$ is indeed the Perron-Frobenius root or spectral radius of $A$. $r(A)$ was shown to be a convex function of a single diagonal element of nonnegative $A$, when all other elements of $A$ are held constant, by use of determinants [1]. $r(A)$ was shown to be a convex function of all diagonal elements jointly of essentially nonnegative $A$ by a Feynman-Kac formula from the theory of random evolutions [2] and by a variational formula of Donsker and Varadhan [3]. Here we give a short proof that $r(A)$ is a convex function of the diagonal of essentially nonnegative $A$ using the Trotter product formula [6] and a theorem on log-convexity due to Kingman [4] as extended by Seneta [5].

The Trotter product formula for any two real $n \times n$ matrices $S$ and $T$ asserts

$$e^{S+T} = \lim_{k \to \infty} \left(e^{S/k}e^{T/k}\right)^k.$$  

A positive function $f$ on a convex domain is log-convex if $\log f$ is convex on that domain. Kingman showed that the set of log-convex functions is closed under positive linear combinations, multiplication, positive powers and lim sup. Let $T(u)$ be a nonnegative (elementwise) $n \times n$ matrix in which each element is either identically 0 or is a log-convex function of a parameter $u$ in some convex domain in $R^m$, $m > 1$. Then, provided that $r(T(u))$ is positive in this domain, $r(T(u))$ is log-convex there [5, p. 83]. (Kingman [4] assumed every element of $T(u)$ positive.)

Let $D$ be a diagonal real $n \times n$ matrix $\text{diag}(d_1, \ldots, d_n)$. Let $A$ be essentially nonnegative.

**Theorem.** $r(A + D)$ is a convex function of $D$. 

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Proof. We have \( r(A + D) = \log e^{r(A+D)} = \log r(e^{A+D}) \). Since \( r \) is a continuous function, (1) gives

\[
r(e^{A+D}) = \lim_{k \to \infty} r\left[ \left( e^{A/k} e^{D/k} \right)^k \right],
\]

hence

\[
r(A + D) = \lim_{k \to \infty} \log r\left[ \left( e^{A/k} e^{D/k} \right)^k \right]. \tag{2}
\]

Now for every positive integer \( k \), every element of \( e^{A/k} \) is nonnegative. Moreover \( r(e^{A/k}) = e^{r(A/k)} > 0 \), and multiplying \( e^{A/k} \) by a nonnegative nonsingular matrix will yield a nonnegative matrix whose spectral radius is also positive. In particular, \( T_k(D) = (e^{A/k} e^{D/k})^k \) is nonnegative with \( r(T_k(D)) > 0 \). For \( i = 1, 2, \ldots, n \), the \( i \)th diagonal element of \( e^{D/k} \), which is just \( e^{d_i/k} \), is a log-convex function of \( D \). Hence every element of \( T_k(D) \) is either 0 or a log-convex function of \( D \). The theorem of Kingman as extended by Seneta implies that \( \log r(T_k(D)) \) is a convex function of \( D \). By (2), \( r(A + D) \) is also a convex function of \( D \).

Some applications of this theorem are discussed in [1].

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References


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