CONVEXITY OF THE DOMINANT EIGENVALUE OF AN ESSENTIALLY NONNEGATIVE MATRIX

JOEL E. COHEN

Abstract. The dominant eigenvalue of a real $n \times n$ matrix $A$ with nonnegative elements off the main diagonal is a convex function of the diagonal of $A$. We give a short proof using Trotter's product formula and a theorem on log-convexity due to Kingman.

An $n \times n$ real matrix $A$ with nonnegative elements $a_{ij}$ ($i \neq j$) off the main diagonal is called essentially nonnegative. Such an $A$ has an eigenvalue $r(A)$, called the dominant eigenvalue, that is real and greater than or equal to the real part of any other eigenvalue of $A$. $r(A)$ need not be the eigenvalue of $A$ with the largest modulus, but when every element of $A$ is nonnegative, $r(A)$ is indeed the Perron-Frobenius root or spectral radius of $A$. $r(A)$ was shown to be a convex function of a single diagonal element of nonnegative $A$, when all other elements of $A$ are held constant, by use of determinants [1]. $r(A)$ was shown to be a convex function of all diagonal elements jointly of essentially nonnegative $A$ by a Feynman-Kac formula from the theory of random evolutions [2] and by a variational formula of Donsker and Varadhan [3]. Here we give a short proof that $r(A)$ is a convex function of the diagonal of essentially nonnegative $A$ using the Trotter product formula [6] and a theorem on log-convexity due to Kingman [4] as extended by Seneta [5].

The Trotter product formula for any two real $n \times n$ matrices $S$ and $T$ asserts

$$e^{S+T} = \lim_{k \to \infty} (e^{S/k} e^{T/k})^k. \quad (1)$$

A positive function $f$ on a convex domain is log-convex if $\log f$ is convex on that domain. Kingman showed that the set of log-convex functions is closed under positive linear combinations, multiplication, positive powers and lim sup. Let $T(u)$ be a nonnegative (elementwise) $n \times n$ matrix in which each element is either identically 0 or is a log-convex function of a parameter $u$ in some convex domain in $\mathbb{R}^m$, $m > 1$. Then, provided that $r(T(u))$ is positive in this domain, $r(T(u))$ is log-convex there [5, p. 83]. (Kingman [4] assumed every element of $T(u)$ positive.)

Let $D$ be a diagonal real $n \times n$ matrix $\text{diag}(d_1, \ldots, d_n)$. Let $A$ be essentially nonnegative.

Theorem. $r(A + D)$ is a convex function of $D$. 

Received by the editors December 12, 1979 and, in revised form, July 30, 1980.


Key words and phrases. Perron-Frobenius root, convexity, log-convexity, Trotter product formula, spectral radius.

Supported in part by U. S. National Science Foundation grant DEB80-11026.
Proof. We have $r(A + D) = \log e^{r(A + D)} = \log r(e^{A + D})$. Since $r$ is a continuous function, (1) gives

$$r(e^{A + D}) = \lim_{k \to \infty} r\left( (e^{A/k} e^{D/k})^k \right),$$

hence

$$r(A + D) = \lim_{k \to \infty} \log r\left( (e^{A/k} e^{D/k})^k \right). \quad (2)$$

Now for every positive integer $k$, every element of $e^{A/k}$ is nonnegative. Moreover $r(e^{A/k}) = e^{r(A/k)} > 0$, and multiplying $e^{A/k}$ by a nonnegative nonsingular matrix will yield a nonnegative matrix whose spectral radius is also positive. In particular, $T_k(D) = (e^{A/k} e^{D/k})^k$ is nonnegative with $r(T_k(D)) > 0$. For $i = 1, 2, \ldots, n$, the $i$th diagonal element of $e^{D/k}$, which is just $e^{a_i/k}$, is a log-convex function of $D$. Hence every element of $T_k(D)$ is either 0 or a log-convex function of $D$. The theorem of Kingman as extended by Seneta implies that $\log r(T_k(D))$ is a convex function of $D$. By (2), $r(A + D)$ is also a convex function of $D$.

Some applications of this theorem are discussed in [1].

I thank E. Seneta and a referee for corrections and comments.

References


Department of Populations, Rockefeller University, New York, New York 10021