# ON THE COLLATZ $3 \boldsymbol{n}+1$ ALGORITHM 

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Abstract. The number theoretic function $s(n)=\frac{1}{2} n$ if $n$ is even, $s(n)=3 n+1$ if $n$ is odd, generates for each $n$ a Collatz sequence $\left\{s^{k}(n)\right\}_{k=0}^{\infty}, s^{0}(n)=n, s^{k}(n)=$ $s\left(s^{k-1}(n)\right)$. It is shown that if a Collatz sequence enters a cycle other than the 4,2 , $1,4, \ldots$ cycle, then the cycle must have many thousands of terms.

1. Introduction. The Collatz $3 n+1$ algorithm is defined as a function $s: N \rightarrow N$ on the set of positive integers by

$$
s(n)= \begin{cases}n / 2 & \text { if } n \text { is even } \\ 3 n+1 & \text { if } n \text { is odd }\end{cases}
$$

Let $s^{0}(n)=n$ and $s^{k}(n)=s\left(s^{k-1}(n)\right)$ for $k \in N$. The Collatz sequence for $n$ is

$$
C(n)=\left\{s^{k}(n)\right\}_{k=0}^{\infty} .
$$

For example, $C(17)=\{17,52,26,13,40,20,10,5,16,8,4,2,1,4, \ldots\}$.
The original problem of Collatz concerns the existence of cycles in Collatz sequences. It is conjectured that every Collatz sequence ends in the cycle $4,2,1,4, \ldots$ So many people have worked on this problem in the nearly fifty years of its existence that it is almost part of mathematical folklore. Martin Gardner reports [2] that the conjecture has been verified for all $n<60,000,000$; Riho Terras says that the conjecture has been verified for all $n<2,000,000,000$.

The only published results to date [1], [3] are probabilistic in nature, and tend to strengthen belief in the conjecture.

This paper proves that there are no other "short" cycles: if a cycle exists which does not contain 1 , then it has many thousands of terms.
2. Stopping time. Collatz's conjecture is equivalent to the conjecture that for each $n \in N, n>1$, there exists $k \in N$ such that $s^{k}(n)<n$. The least $k \in N$ such that $s^{k}(n)<n$ is called the stopping time of $n$, which we will denote by $\sigma(n)$.

It is not hard to verify that
$\sigma(n)=1$ if $n$ is even,
$\sigma(n)=3$ if $n \equiv 1(\bmod 4)$,
$\sigma(n)=6$ if $n \equiv 3(\bmod 16)$,
$\sigma(n)=8$ if $n \equiv 11$ or $23(\bmod 32)$,
$\sigma(n)=11$ if $n \equiv 7,15$, or $59(\bmod 128)$,
$\sigma(n)=13$ if $n \equiv 39,79,95,123,175,199$, or $219(\bmod 256)$,
and so forth. Everett [1] proves that almost all $n \in N$ have finite stopping time, and Terras [3] gives a probability distribution function for stopping times. Most positive

[^0]integers have small stopping times; the above list accounts for $237 / 256 \approx 93 \%$ of them. However, stopping times can be arbitrarily large, for $\sigma\left(2^{n}-1\right)>2 n$. Some interesting cases of larger stopping times are $\sigma(27)=96, \sigma(703)=132, \sigma(35,655)=$ $220, \sigma(270,271)=311$, and $\sigma(1,027,341)=347$. In a computation of stopping times of integers up to $1,065,000$, the largest observed stopping time was 347 .
3. A term formula. Let $C^{k}(n)$ consist of the first $k$ terms of the Collatz sequence for $n$. Let $m$ (which depends on $n$ and $k$ ) be the number of odd terms in $C^{k}(n)$, and let $d_{i}$ be the number of consecutive even terms immediately following the $i$ th odd term. Let $d_{0}$ be the number of even terms preceding the first odd term. Then the next term in the Collatz sequence for $n$ is
\[

$$
\begin{equation*}
s^{k}(n)=\frac{3^{m}}{2^{k-m}} n+\sum_{i=1}^{m} \frac{3^{m-i}}{2^{d_{i}+\cdots+d_{m}}} . \tag{1}
\end{equation*}
$$

\]

Note that $k-m=d_{0}+d_{1}+\cdots+d_{m}$.
4. Coefficient stopping time. By the coefficient of $s^{k}(n)$ is meant the coefficient of $n$ in (1), namely $3^{m} / 2^{k-m}$. The coefficient stopping time of $n$ is the least $k \in N$ such that the coefficient of $s^{k}(n)$ is less than 1 , and is denoted by $\kappa(n)$. Thus $\kappa(n)$ is the least $k$ such that $3^{m}<2^{k-m}$.

It is clear that if $s^{k}(n)<n$, then the coefficient of $s^{k}(n)$ is less than 1 ; thus $\kappa(n) \leqslant \sigma(n)$ for all $n \in N, n>1$. We conjecture that $\kappa(n)=\sigma(n)$, and have verified it for all $n \leqslant 1,150,000$.

For $m=0$ or $m \in N$, let $p(m)=\left[m \log _{2} 3\right]$; then $2^{p(m)} \leqslant 3^{m}<2^{1+p(m)}$. Thus if $\kappa(n)=k$, then $k-m=1+p(m)$, or $k=m+1+p(m)$. These, then, are the possible coefficent stopping times.

We can also identify those $n \in N$ with a given coefficient stopping time. Clearly, $\kappa(n)=1$ if and only if $n$ is even. If $k=m+1+p(m)>1$ is a possible coefficient stopping time, let $d_{0}=0$ and $d_{1}, d_{2}, \ldots, d_{m} \in N$ be such that $d_{1}+\cdots+d_{i} \leqslant$ $p(i)$ for $i=1,2, \ldots, m-1$, and $d_{1}+\cdots+d_{m}=p(m)+1$. Let $x$ and $y$ be integers such that $3^{m} x+2^{1+p(m)} y=1$. Then $\kappa(n)=k$ if and only if

$$
n \equiv-x\left(\sum_{i=1}^{m} 3^{m-i} 2^{d_{0}+\cdots+d_{i-1}}\right) \quad\left(\bmod 2^{1+p(m)}\right)
$$

We are able to prove that $\kappa(n)=\sigma(n)$ under certain bound conditions which arise from a study of the powers of 2 and the powers of 3.
5. Powers of 2 and 3. The behavior of a Collatz sequence is clearly related to the way in which the powers of 2 are distributed among the powers of 3 . We were surprised to find that the powers of 2 appear to be bounded away from the powers of 3 by an amount which grows almost as rapidly as the power of 3 .

To be specific, we choose $M \in N$, and let

$$
b(M)=\max _{j<M}\left\{-\log _{3}\left(1-2^{p(j)} 3^{-j}\right)\right\}
$$

and

$$
B(M)=\max _{j<M}\left\{-\log _{3}\left(2^{1+p(j)} 3^{-j}-1\right)\right\}
$$

Then it follows that

$$
\begin{equation*}
3^{m}-2^{p(m)}>3^{m-b(M)} \tag{2}
\end{equation*}
$$

and

$$
\begin{equation*}
2^{1+p(m)}-3^{m}>3^{m-B(M)} \tag{3}
\end{equation*}
$$

for all $m \leqslant M$. The values of $M$ at which $b(M)$ and $B(M)$ increase are given in Table 1, with the corresponding values of $b$ and $B$. (These calculations were carried out on a Hewlett-Packard HP-19C programmable calculator.)

| $M$ | $b(M)$ | $B(M)$ | $M$ | $b(M)$ | $B(M)$ | $M$ | $b(M)$ | $B(M)$ |
| :---: | :--- | :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 1 | 306 |  | 6.267 | 8286 | 6.9 |  |
| 2 | 2 |  | 359 | 6.23 |  | 8951 | 7.0 |  |
| 3 |  | 1.535 | 665 | 9.14 |  | 9616 | 7.1 |  |
| 5 |  | 2.665 | 971 |  | 6.31 | 10281 | 7.2 |  |
| 7 | 2.508 |  | 1636 |  | 6.35 | 10946 | 7.3 |  |
| 12 | 3.921 |  | 2301 |  | 6.39 | 11611 | 7.4 |  |
| 17 |  | 2.946 | 2966 |  | 6.44 | 12276 | 7.6 |  |
| 29 |  | 3.346 | 3631 |  | 6.49 | 12941 | 7.8 |  |
| 41 |  | 4.062 | 4296 |  | 6.53 | 13606 | 8.0 |  |
| 53 | 5.618 |  | 4961 |  | 6.59 | 14271 | 8.4 |  |
| 94 |  | 4.246 | 5626 |  | 6.65 | 14936 |  | 8.9 |
| 147 |  | 4.477 | 6291 |  | 6.71 | 15601 |  | 10.2 |
| 200 |  | 4.785 | 6956 |  | 6.77 | 16266 | 9.4 |  |
| 253 |  | 5.253 | 7621 |  | 6.8 | 31867 | 9.9 |  |

Table 1. Values at which $b(M)$ and $B(M)$ increase
6. The main theorem. Now we can prove that $\kappa(n)=\sigma(n)$ if the number of odd terms encountered in the Collatz sequence is not too great. The following theorem makes the bound condition precise.

Theorem. In the notation of (1), if $\kappa(n)=k$ and

$$
m<\min \left\{M,(n / 2) 3^{1-B(M)}\left(1-3^{-b(M)}\right)^{-1}\right\}
$$

then $\sigma(n)=\kappa(n)$.
Proof. If $\kappa(n)=k$, then $3^{m} / 2^{k-m}<1$ and

$$
\frac{3^{m-i}}{2^{d_{i}+\cdots+d_{m}}}<\frac{2^{d_{0}+\cdots+d_{i-1}}}{3^{i}}<\frac{2^{p(i-1)}}{3^{i}}<\frac{1}{3}\left(1-3^{-b(M)}\right)
$$

by (2). Thus

$$
s^{k}(n)<\frac{3^{m}}{2^{k-m}} n+\frac{m}{3}\left(1-3^{-b(M)}\right)
$$

Now if $s^{k}(n) \geqslant n$, then $n<\left(3^{m} / 2^{k-m}\right) n+(m / 3)\left(1-3^{-b(M)}\right)$, and therefore

$$
n<\frac{2}{3} m\left(1-3^{-b(M)}\right) \frac{2^{p(m)}}{3^{m}} \cdot 3^{B(M)}
$$

by (3). This leads by the hypothesis to $n<n$, a contradiction. Hence $s^{k}(n)<n$, and $\sigma(n) \leqslant k$. But $\kappa(n)<\sigma(n)$, so $\sigma(n)=\kappa(n)$.

It is conjectured that the bound condition in the theorem is unnecessary, or is automatically satisfied so that $\sigma(n)=\kappa(n)$ in all cases.
7. Application to cycles. Suppose a Collatz sequence enters a cycle which does not contain 1 . Let $n$ be the least term of the cycle; then $s^{k}(n)=n$ for some $k \in N$. Hence $\kappa(n) \leqslant k$, so that if $\sigma(n)=\kappa(n)$, then $n$ is not the least term of the cycle after all. Thus if stopping time and coefficient stopping time are always the same, then the only cycle is the $4,2,1, \ldots$ cycle.

The same contradiction arises if the number of odd terms in the cycle satisfies the bound in the theorem. Thus if there is a cycle not containing 1 , the number $m$ of odd terms in the cycle must satisfy $m \geqslant \min \left\{M,(n / 2) 3^{1-B(M)}\left(1-3^{-b(M)}\right)^{-1}\right\}$, where $n$ is the least term of the cycle.

Using $n>60,000,000, M=14,000, B(M)=8.0$, and $b(M)=9.14$, we find $m \geqslant 13,700$, and hence $k \geqslant 35,400$. Thus any cycle not containing 1 must have at least 35,400 terms.

If the number 2 billion turns out to be the lower bound for $n$, as alleged, then $M=41,000, B(M)=10.2$, and $b(M)=9.9$ yield $m>40,700$ and $k>105,000$.

In any event, there is only one "short" cycle, the known one.

## References

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