ON THE DUAL OF A CERTAIN OPERATOR IDEAL

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Abstract. For complex Banach spaces $E$ and $F$ and a real number $1 < p < \infty$ let $S^p(E, F)$ denote the operator ideal obtained by complex interpolation between the nuclear and the compact operators. If $E$ and $F$ are reflexive and one of them has the approximation property the dual of $S^p(E, F)$ is shown to be $S^{p'}(E', F')$, $p'$ conjugate to $p$.

Let $N$ denote the ideal of nuclear and $K$ the ideal of compact linear operators. By complex interpolation between $N$ and $K$ there is associated to every real number $1 < p < \infty$ the operator ideal $S^p$ in [4], i.e., $S^p(E, F) = [N(E, F), K(E, F)]_{1/p}$ for any complex Banach spaces $E$ and $F$ (that this definition gives the same as that of [4, p. 101] follows from [1, 9.3]). It is shown there that for any complex separable Hilbert space $H$, $S^p(H, H)$ consists of those compact operators whose moduli have $p$th power summable eigenvalues. It is then a classical result of Schatten and von Neumann that the Banach space dual of $S^p(H, H)$ may be identified with $S^{p'}(H, H)$. Here we are showing

Proposition. Let $E$ and $F$ be complex Banach spaces and $p$ a real number with $1 < p < \infty$, $1/p' + 1/p = 1$. If $E$ and $F$ are reflexive and one of them has the approximation property then the dual of $S^p(E, F)$ may be identified isometrically with $S^{p'}(E', F')$, the pairing given by $\langle S, T \rangle = \text{trace}(T^* \circ S)$ when $E$ has the approximation property; $\langle T, S \rangle = \text{trace}(S \circ T')$ when $F$ has it.

In particular the space $S^p(E, F)$ is reflexive.

Proof. We shall use the notation of [1] without further explanation. Let us assume that $E$ has the approximation property.

First step. Since $E$ is reflexive $E'$ has it too by [3 Proposition 36.1]. By Satz 3 and Satz 7 in [4], for every $S \in S^p(E', F')$ and $T \in S^p(E, F)$ the product $T' \circ S$ is nuclear, and $\|T' \circ S\|_N \leq \|T''\|_{SP'}\|S\|_{SP'} \leq \|T\|_{SP'}\|S\|_{SP'}$. Since $E'$ has the approximation property, trace($T' \circ S$) is well defined and $\langle T' \circ S \rangle < \|T' \circ S\|_N < \|T\|_{SP'}\|S\|_{SP'}$, such that $\beta: S^p(E', F') \to S^p(E, F)$, given by $\langle T, \beta S \rangle = \text{trace}(T' \circ S)$, is a linear contraction.

Second step. $\beta$ is an isometry. Since the linear mappings of finite rank are dense in $S^p(E', F')$ it suffices to show that

$\|S\|_{S^p(E', F')} < \|\beta S\|_{S^p(E, F)}$
for any such map. So let \( S: E' \to F' \) be a linear map of finite rank. By the theorem of Hahn-Banach there exists a linear form \( L \) in \( S'(E', F')' \) of norm 1 with \( \|S\|_{S'} = \langle S, L \rangle = \text{trace}(L' \circ S) \), when we identify \( L \) with the corresponding bounded linear map from \( E \) to \( F \). By definition and the duality theorem 12.1 in [1] one has

\[
S'(E', F') = \left\{ N(E', F'), K(E', F') \right\}^{1/p}
\]

since \( N(E', F') \) is dense in \( K(E', F') \) because of the approximation property of \( E'' = E \). Identifying \( N(E', F')' = (E'' \otimes F')' = H(E, F) \), the space of bounded linear mappings from \( E \) to \( F \), and \( K(E', F') = (E'' \otimes F')' = I(E, F) \), the space of integral mappings from \( E \) to \( F \), which coincides with \( N(E, F) \), since \( F \) is reflexive [3, Théorème 10.1], we obtain

\[
S'(E', F') = \left\{ N(E, F), H(E, F) \right\}'
\]

with \( t = 1 - 1/p = 1/p' \).

So for every \( \varepsilon > 0 \) we can find a function \( h \) in \( \mathcal{F}(N(E, F), H(E, F)) \) with \( \|h\|_{\mathcal{F}} < 1 + \varepsilon \) and whose derivative \( h'(t) \) at the point \( t \) equals \( L \).

For this function \( h \) we construct in the usual manner (cf. [2]) a sequence of functions \( h_n \) in \( \mathcal{F}(N(E, F), H(E, F)) \) with \( \lim_{n \to \infty} h_n(t) = h'(t) \); for example,

\[
h_n(z) = \exp(z^2/n)[h(z + i/n) - h(z)]/i \quad \text{for } 0 < \Re z < 1, n > 1.
\]

\[\|h_n\|_{\mathcal{F}} < e^{1/n}\|h\|_{\mathcal{F}} \] for every \( n \). By 9.3 in [1] (second line from below) we have \( \mathcal{F}(N(E, F), H(E, F)) = \mathcal{F}(N(E, F), K(E, F)) \) isometrically so that \( h_n(t) \in [N(E, F), K(E, F)]_t = S'(E, F) \) and \( \|h_n(t)\|_{S'} < \|h_n\|_{\mathcal{F}} < e^{1/n}(1 + \varepsilon) \) for every \( n \).

All together one has

\[
\|S\|_{S'(E', F')} = \text{trace}(L' \circ S) = \text{trace}\left(\left[ h'(t) \right]' \circ S \right)
\]

\[
= \lim \text{trace}\left(\left[ (h(t + i/n) - h(t))n/i \right]' \circ S \right)
\]

\[
= \lim \text{trace}(\exp(-t^2/n)[h_n(t)]' \circ S)
\]

\[
= \lim \exp(-t^2/n)\langle h_n(t), BS \rangle
\]

\[
< \lim \sup \exp(-t^2/n)\|h_n(t)\|_{S'}\|BS\|_{S'(E, F)'}
\]

\[
< \lim \sup \exp\left(1 - t^2/n\right)(1 + \varepsilon)\|BS\|_{S'(E, F)'}
\]

\[< (1 + \varepsilon)\|BS\|_{S'(E, F)'}.
\]

i.e.,

\[
\|S\|_{S'(E', F')} < \|BS\|_{S'(E, F)'}.
\]

Since this inequality obtains for every \( S \in S'(E', F') \), the isometry of \( \beta \) follows in conjunction with the contractivity of \( \beta \).

**Third step.** The image of \( \beta \) is dense in \( S'(E, F)' \). First of all note that \( S'(E, F)' \) is equal to \( [N(E, F), K(E, F)]^{1/p'} = [N(E', F'), H(E', F')]^{1/p'} \) by the duality theorem (cf. Second step). Now by the last formula in 9.3 of [1] this last space is equal to \( [N(E', F'), K(E', F')]^{1/p'} \) which by definition is contained in \( N(E', F') + K(E', F') = K(E', F') \). Since \( K(E', F') \) is the closure of the operators of finite rank and these are contained in \( S'(E', F') \), \( \beta \) has dense image.
So $\beta: S^p(E', F') \rightarrow S^p(E, F')$ is an isometric isomorphism. Since the proof works as well when $F$ (hence $F'$) has the approximation property the proposition is established.

Let now $G$ denote a compact group with normalized Haar measure and $L^q(G)$ its complex Lebesgue spaces. Let $S^p_G(L^q(G), L^q(G))$ denote the set of those operators in $S^p(L^q(G), L^q(G))$ which are (left-or-right-) translation invariant under $G$. Since it is a closed subspace of $S^p(L^q(G), L^q(G))$ one arrives at

**Corollary.** Let $G$ be a compact group and $1 < p, q < \infty$. Then the two-sided $q$-Segal algebras $S^p_G(L^q(G), L^q(G))$ are reflexive Banach spaces.

The assertion of the corollary remains true for $q = 1$, since then

$$S^p_G(L^1(G), L^1(G)) = L^p(G),$$

which was the starting point for this note (cf. [5]).

**Acknowledgement.** I am much indebted to Dr. Michael Cwikel from Technion Haifa for making me familiar with the technique used in Second step and for kindly sending me a proof of the last formula of 9.3 in [1], communicated privately to him by A. P. Calderón. The proof is reproduced in J. Bergh, *On the relation between the two complex methods of interpolation*, to appear in Indiana Univ. Math. J.

**References**


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