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LOCAL SOLVABILITY OF CONSTANT COEFFICIENT PARTIAL DIFFERENTIAL EQUATIONS AS A SMALL DIVISOR PROBLEM

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ABSTRACT. Local solvability of $P(D)u = f$ is reduced to a small divisor problem in Fourier series.

The existence of a fundamental solution for an arbitrary constant coefficient operator $P(D)$ was established by Malgrange [3] and Ehrenpreis [1], and Hörmander [2] obtained fundamental solutions with the best regularity properties. The latter result shows that there is a solution u to $P(D)u = f \in L^2_{\text{loc}}$ with u and $P^{(\alpha)}(D)u$ in L^2_{loc}, where $P^{(\alpha)}(\xi) = D^\alpha_\xi P(\xi)$. In this note we solve the equation $P(D)u = f \in C_0^\infty(\Omega)$ or $\mathcal{E}'(\Omega)$ on an arbitrary bounded open set Ω in $\mathbf{R}^n$ by reducing it to a division problem in Fourier series which is solved by quite elementary means. As a simple remark at the end we note that if the coefficients of $P(D)$ are rational this Fourier series method can also produce $u \in L^2$ if $f \in L^2(\Omega)$.

Let $P(D)$ be a polynomial of degree m in $D_1, \ldots, D_n$ where $D_j = (1/i)(\partial/\partial x_j)$. We first note that to solve $P(D)u = f$ is equivalent to solving $P(D + \alpha)v = g$ where $\alpha \in \mathbf{R}^n$ is fixed and

$$g = e^{-i\alpha\cdot x}f, \qquad v = e^{-i\alpha\cdot x}u.$$

Suppose that g is supported in the unit ball. Thinking of g now as a function on the torus $\mathbf{T}^n = \mathbf{R}^n/(2\pi\mathbf{Z})^n$, we may get our solution v locally on $\mathbf{R}^n$ by solving $P(D + \alpha)v = g$ globally on $\mathbf{T}^n$ for an appropriate α. We therefore prove

THEOREM 1. *For almost all* $\alpha \in A = \{\alpha \in \mathbf{R}^n: 0 \leqslant \alpha_j \leqslant 1\}$,

$$P(D + \alpha): C^\infty(\mathbf{T}^n) \to C^\infty(\mathbf{T}^n)$$

is an isomorphism, as is

$$P(D + \alpha): \mathcal{D}'(\mathbf{T}^n) \to \mathcal{D}'(\mathbf{T}^n).$$

Since C^∞ functions on the torus are characterised by having rapidly decreasing Fourier coefficients and distributions on the torus are characterised by having Fourier coefficients which are polynomially bounded, Theorem 1 is an immediate consequence of

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PROPOSITION 2. *There is a* $d \in \mathbf{R}$ *such that for almost all* $\alpha \in A$,

$$|P(k+\alpha)^{-1}| < C_\alpha(1+|k|^2)^d \quad \textit{for all } k \in \mathbf{Z}^n. \tag{1}$$

PROOF. Suppose we can show that for some d, C, the set

$$S_C = \left\{\xi \in \mathbf{R}^n : |P(\xi)^{-1}| > C(1+|\xi|^2)^d\right\} \tag{2}$$

has finite Lebesgue measure; $\operatorname{meas}(S_C) < \infty$. Then, by the Lebesgue dominated convergence theorem $\lim_{a\to\infty} \operatorname{meas}(S_a) = 0$, and thus given $1 > \varepsilon > 0$ the set $\pi(S_a)$ in $\mathbf{R}^n/\mathbf{Z}^n \leftarrow \mathbf{R}^n$: π has measure smaller than ε if a is large enough. Therefore, for such a, the set $A_a = \{\alpha \in A : \pi(S_a + \alpha) \cap \pi(\mathbf{Z}^n) = \varnothing\}$ has measure larger than $1-\varepsilon$, and for $\alpha \in A_a$ (1) holds.

To establish (2), it is enough to show that, for some $\delta > 0$,

$$\int_{|\xi|>1} |P(\xi)|^{-\delta}|\xi|^{-2n}\, d\xi < \infty. \tag{3}$$

A change of variable gives

$$\int_{|\xi|>1} |P(\xi)|^{-\delta}|\xi|^{-2n}\, d\xi = \int_{|\xi|<1} \left|P\left(\frac{\xi}{|\xi|^2}\right)\right|^{-\delta} d\xi$$
$$= \int_{|\xi|<1} |\xi|^{2m\delta}|Q(\xi)|^{-\delta}\, d\xi$$

with $Q(\xi) = |\xi|^{2m}P(\xi/|\xi|^2)$ a polynomial of degree $2m$. Thus the following lemma establishes (3) and finishes the proof of Proposition 2.

LEMMA 3. *Let* $Q(\xi)$ *be a polynomial of degree* r. *Then, for* $0 < \delta < 1/r$, $|Q(\xi)|^{-\delta} \in L^1_{\mathrm{loc}}$.

PROOF. We may as well assume that $Q(0) = 0$ and show that $|Q|^{-\delta}$ is integrable near the origin. Now $Q(\xi)$ restricted to some line passing through the origin must be a polynomial of degree r in one variable. Changing coordinates, we may assume that it is the line $\xi' = (\xi_2, \ldots, \xi_n) = 0$. Thus $Q(\xi) = c\xi_1^r + p_{r-1}(\xi')\xi_1^{r-1} + \cdots + p_0(\xi')$, $c \neq 0$. If $0 < \delta < 1/r$ an easy one variable calculation shows that $\int_{-1}^1 |Q(\xi_1, \xi')|^{-\delta}\, d\xi_1 <$ const for all $|\xi'| < 1$. Q.E.D.

REMARK 1. Using dilations of $\mathbf{R}^n$ and Theorem 1 shows that we can solve $P(D)u = f \in C_0^\infty(\Omega)$ or $\mathcal{E}'(\Omega)$ on an arbitrary bounded open set $\Omega \subset \mathbf{R}^n$.

REMARK 2. Proposition 2 shows that given a polynomial $P(\xi)$ one can find an affine lattice on which P^{-1} is polynomially bounded. If P has integer (or more generally, rational) coefficients, it is very easy to produce an affine lattice on which P^{-1} is bounded. Indeed, if $P(\xi) = \sum_{|\beta|<m} a_\beta \xi^\beta$, pick $q = (q_1, \ldots, q_n) \in \mathbf{Z}^n$, let $p = (q_1 \ldots q_n)^m$, and consider

$$Q(k) = P\left(pk_1 + \frac{1}{q_1}, pk_2 + \frac{1}{q_2}, \ldots\right) = \sum_{\beta\neq 0} b_\beta(q)k^\beta + \sum_{|\beta|<m} a_\beta \frac{1}{q^\beta} \tag{4}$$

with $b_\beta(q) \in \mathbf{Z}$. By choosing q appropriately we can make sure that the second sum on the right of (4) is not an integer, and therefore $|Q(k)|^{-1}$ is bounded on $\mathbf{Z}^n$. This

then gives an alternate proof of local solvability of equations with rational coefficients, and gives a solution $u \in L^2(\Omega)$ to $P(D)u = f \in L^2(\Omega)$ in this case, for Ω bounded.

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