COMMENTS ON NOTES OF STANOJEVIĆ ET AL.

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Abstract. Some further inequalities which characterize inner-product spaces among normed linear spaces are derived by averaging characters of locally compact groups.


\[(S, \sim) \quad \text{For all } x, y \in S, \quad 2^{-1}(\|x + y\|^2 + \|x - y\|^2) \sim 2, \text{ in words:}\]

the average of \(\|x + zy\|^2\) over the group \(\{1, -1\}\) \(\sim 2,\)

of inner-product spaces (see Day, Normed linear spaces, Chapter VII, §3, for a general view of such characterizations) suggested that a modification of the proof of the Dvoretzky-Rogers Theorem (op. cit., p. 80, Lemma 1) would prove a family of “averaging” characterizations of inner-product spaces. For notation, \(\Gamma\) is the circle group in the complex plane, \(S\) is the unit sphere \(\{x||x|| = 1\}\) in a normed space \(N\), and \(\sim\) means one of the relations \(<, =, \text{ or } >\) but the same one for all \(x\) and \(y\) in \(S\).

Theorem A. Let \(G \neq 1\) be a closed subgroup of the circle group \(\Gamma\) and let \(H\) be Haar measure on \(G\) normalized so that \(\mathcal{H}(G) = 1\). Let

\[\Psi_G(x, y) = \int_G \|x + zy\|^2 \, dH(z).\]

Then a complex-linear normed space \(N\) is an inner-product space if and only if

\[(\Psi_G, \sim) \quad \text{for all } x, y \in S, \Psi_G(x, y) \sim 2.\]

The note of Stanojević and Suchanek (Proc. Amer. Math. Soc. 81 (1981), 101–103) gives a more general result. Their paper does not, however, suggest the finite examples which are prominent in Theorem A because the only closed proper subgroups of \(\Gamma\) are the groups \(G_k\) of \(k\)th roots of unity. If \(\theta\) is a principal \(k\)th root of unity, then \(N\) is an inner-product space if and only if for each (or for one) \(k > 2\)

\[(\Psi_k, \sim) \quad \text{for each } x \text{ and } y \in S, \quad k^{-1} \sum_{p < k} \|x + \theta^p y\|^2 \sim 2.\]

The case \(k = 2\) is \((S, \sim)\). The case \(k = 3\) already gives an unfamiliar condition

\[\left(\|x + y\|^2 + \|x + \omega y\|^2 + \|x + \omega^2 y\|^2\right)/3 \sim 2,\]

where, of course, \(\omega^3 = 1\).
A result similar to Theorem A gives an averaging analog of the condition of Penico and Stanojević; replace \( \Psi_G \) above by

\[
\Phi_G(x, y) = \int_G \| (\text{Re}(z))x + (\text{Im}(z))y \|^2 \, dH(z).
\]

One can make a corresponding analog of the theorem of Stanojević and Suchanek by replacing their integrand by \( \| (\text{Re}(\gamma(g)))x + (\text{Im}(\gamma(g)))y \|^2 \).

2. Von Neumann (Trans. Amer. Math. Soc. 36 (1934), p. 445) pointed out that the space \( AP(G) \) of almost periodic functions on an arbitrary group \( G \) is a natural generalization of the space of continuous functions on a compact group. One aspect of this is the existence on \( AP(G) \) of a unique invariant mean \( m \), which specializes to a normalized Haar measure when \( G \) is compact. Since each character of \( G \) is almost periodic, this point of view allows a generalization of the theorem of Stanojević and Suchanek sufficiently broad to allow as a corollary the generalization of Theorem A to all subgroups of \( \Gamma \).

**Theorem B.** A complex-linear space \( N \) is an inner-product space if there exists a topological group \( G \) and a continuous character \( \gamma \neq 1 \) of \( G \) such that

\[
(\Psi_{\gamma, \sim}) \quad \text{for all } x, y \in S, \quad \Psi_{\gamma}(x, y) = m_g(\| x + \gamma(g)y \|^2) \sim 2.
\]

Conversely, if \( N \) is an inner-product space, \( (\Psi_{\gamma, =}) \) holds for all \( G \) and all \( \gamma \neq 1 \).

The proof is that of Theorem A or of the theorem of Stanojević and Suchanek.

Since the invariant mean for the almost periodic functions on the integers or on the reals can be calculated as a limit of averages, special cases of Theorem B say that \( N \) is an inner-product space if and only if for each (or for one) \( z \neq 1 \) in \( \Gamma \) and for all \( x, y \) in \( S \)

(i) \[
\lim_{K \to \infty} K^{-1} \sum_{p < K} \| x + z^p y \|^2 \sim 2,
\]

or

(ii) \[
\lim_{K \to \infty} (2K)^{-1} \int_{-K}^{K} \| x + \exp(it)y \|^2 \, dt \sim 2.
\]

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