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COMMENTS ON NOTES OF STANOJEVIĆ ET AL.

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ABSTRACT. Some further inequalities which characterize inner-product spaces among normed linear spaces are derived by averaging characters of locally compact groups.

1. Comparing results in Penico and Stanojević (Proc. Amer. Math. Soc. **79** (1980), 427–430) with the Schoenberg characterization:

$$(S, \sim) \quad \textit{For all } x, y \textit{ in } S,\ 2^{-1}(\|x+y\|^2 + \|x-y\|^2) \sim 2, \text{ in words:}$$

$$\textit{the average of } \|x+zy\|^2 \textit{ over the group } \{1, -1\} \sim 2,$$

of inner-product spaces (see Day, *Normed linear spaces*, Chapter VII, §3, for a general view of such characterizations) suggested that a modification of the proof of the Dvoretsky-Rogers Theorem (*op. cit.*, p. 80, Lemma 1) would prove a family of "averaging" characterizations of inner-product spaces. For notation, Γ is the circle group in the complex plane, S is the unit sphere $\{x \mid \|x\| = 1\}$ in a normed space N, and $\sim$ means one of the relations $<$, $=$, or $>$ but the same one for all x and y in S.

THEOREM A. *Let $G \neq 1$ be a closed subgroup of the circle group Γ and let H be Haar measure on G normalized so that $H(G) = 1$. Let*

$$\Psi_G(x, y) = \int_G \|x + zy\|^2 \, dH(z).$$

Then a complex-linear normed space N is an inner-product space if and only if

$$(\Psi_G, \sim) \qquad \textit{for all } x, y \textit{ in } S,\ \Psi_G(x, y) \sim 2.$$

The note of Stanojević and Suchanek (Proc. Amer. Math. Soc. **81** (1981), 101–103) gives a more general result. Their paper does not, however, suggest the finite examples which are prominent in Theorem A because the only closed proper subgroups of Γ are the groups G_k of kth roots of unity. If θ is a principal kth root of unity, then N is an inner-product space if and only if for each (or for one) $k > 2$

$$(\Psi_k, \sim) \qquad \textit{for each } x \textit{ and } y \textit{ in } S,\ k^{-1} \sum_{p<k} \|x + \theta^p y\|^2 \sim 2.$$

The case $k = 2$ is $(S, \sim)$. The case $k = 3$ already gives an unfamiliar condition

$$(\|x+y\|^2 + \|x+\omega y\|^2 + \|x+\omega^2 y\|^2)/3 \sim 2,$$

where, of course, $\omega^3 = 1$.

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A result similar to Theorem A gives an averaging analog of the condition of Penico and Stanojević; replace Ψ_G above by

$$\Phi_G(x, y) = \int_G \|(\mathrm{Re}(z))x + (\mathrm{Im}(z))y\|^2 \, dH(z).$$

One can make a corresponding analog of the theorem of Stanojević and Suchanek by replacing their integrand by $\|(\mathrm{Re}(\gamma(g)))x + (\mathrm{Im}(\gamma(g)))y\|^2$.

2. Von Neumann (Trans. Amer. Math. Soc. **36** (1934), p. 445) pointed out that the space $AP(G)$ of almost periodic functions on an arbitrary group G is a natural generalization of the space of continuous functions on a compact group. One aspect of this is the existence on $AP(G)$ of a unique invariant mean m, which specializes to a normalized Haar measure when G is compact. Since each character of G is almost periodic, this point of view allows a generalization of the theorem of Stanojević and Suchanek sufficiently broad to allow as a corollary the generalization of Theorem A to *all* subgroups of Γ.

THEOREM B. *A complex-linear space N is an inner-product space if there exists a topological group G and a continuous character $\gamma \not\equiv 1$ of G such that*

$$(\Psi_\gamma, \sim) \qquad \textit{for all } x, y \textit{ in } S, \ \Psi_\gamma(x, y) = m_g(\|x + \gamma(g)y\|^2) \sim 2.$$

Conversely, if N is an inner-product space, $(\Psi_\gamma, =)$ holds for all G and all $\gamma \not\equiv 1$.

The proof is that of Theorem A or of the theorem of Stanojević and Suchanek.

Since the invariant mean for the almost periodic functions on the integers or on the reals can be calculated as a limit of averages, special cases of Theorem B say that N is an inner-product space if and only if for each (or for one) $z \neq 1$ in Γ and for all x, y in S

$$\text{(i)} \qquad \lim_{K\to\infty} K^{-1} \sum_{p<K} \|x + z^p y\|^2 \sim 2,$$

or

$$\text{(ii)} \qquad \lim_{K\to\infty} (2K)^{-1} \int_{-K}^{K} \|x + \exp(it)y\|^2 \, dt \sim 2.$$

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