A CONTINUOUS VERSION OF THE BORSUK-ULAM THEOREM

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Abstract. Let $p: E \rightarrow B$ be an $n$-sphere bundle, $q: V \rightarrow B$ be an $\mathbb{R}^n$-bundle and $f: E \rightarrow V$ be a fibre preserving map over a paracompact space $B$. Let $\tilde{p}: \tilde{E} \rightarrow B$ be the projectivized bundle obtained from $p$ by the antipodal identification and let $\tilde{A}_f$ be the subset of $\tilde{E}$ consisting of pairs $(e, -e)$ such that $fe = f(-e)$. If the cohomology dimension $d$ of $B$ is finite then the map $(\tilde{p}_! \tilde{A}_f)^*: H^d(B; \mathbb{Z}) \rightarrow H^d(\tilde{A}_f; \mathbb{Z})$ is injective for a continuous cohomology theory $H^*$. Moreover, if the $j$th Stiefel-Whitney class of $q$ is zero for $1 < j < r$ then $(\tilde{p}_! \tilde{A}_f)^*$ is injective in degrees $i > d - r$. If all the Stiefel-Whitney classes of $q$ are zero then $(\tilde{p}_! \tilde{A}_f)^*$ is injective in every degree.

Introduction. The Borsuk-Ulam theorem [1] says that if $f: S^n \rightarrow \mathbb{R}^n$ is a map then the set $A_f$ of points $x \in S^n$ such that $fx = f(-x)$ is nonempty. Because $A_f$ is symmetric with respect to the antipodal involution, it is more convenient to consider the subset $\tilde{A}_f$ of the real projective $n$-space $P^n$ corresponding to $A_f$ under the antipodal identification.

If a single $S^n$ and an $\mathbb{R}^n$ are replaced by continuous families $E \rightarrow B$ with fibre $S^n$ and $V \rightarrow B$ with fibre $\mathbb{R}^n$ over a space $B$, and if $f$ is replaced by a fibre preserving map $f: E \rightarrow V$, one may expect the existence of a cross-section of sorts in the set $A_f$ of pairs $(e, -e)$ such that $e \in E$ and $fe = f(-e)$, at least on an algebraic level.

A result in this direction in the case when $E$ is the product bundle $E = S^k \times S^n$ and $V$ is a single $\mathbb{R}^n$ follows from a theorem proved by J. E. Connett [2]. In this note we are going to consider this question for fibre preserving maps $E \rightarrow V$ where $E$ is an $n$-sphere bundle and $V$ is an $n$-dimensional real vector space bundle over a paracompact space $B$. If $B$ is a point, then the theorem proved below reduces to the classical Borsuk-Ulam theorem.

Main result. If $X$ is a space with an involution $t: X \rightarrow X$, we denote by $\overline{X}$ the orbit space $X/t$ of $t$. If $p: E \rightarrow B$ is a fibre bundle with a fibre preserving involution $t: E \rightarrow E$, we write $\overline{p}: \overline{E} \rightarrow B$ for the bundle $p/t: E/t \rightarrow B$; its fibre is $\overline{X}$, where $X$ is the fibre of $p$. Thus if $p: E \rightarrow B$ is an $n$-sphere bundle, then $\overline{p}: \overline{E} \rightarrow B$ is the associated real projective $n$-sphere bundle.

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If $E$ is any space with an involution $t: E \to E$ and $f: E \to V$ is a map of $E$ into some space $V$, let $A_f$ denote the set of points $e \in E$ such that $fe = fte$ and let $\tilde{A}_f$ be the image of $A_f$ in $\tilde{E}$.

We are going to use the Alexander-Spanier cohomology theory $H^* \mod 2$. The coefficient group $\mathbb{Z}_2$ will be suppressed from the notation. If $Z$ is a space, $A$ is a subset of $Z$ and $i: A \to Z$ is the inclusion map, then the image of a cohomology class $z \in H^*(Z)$ under the induced homomorphism $i^*: H^*(Z) \to H^*(A)$ will sometimes be denoted by $z|A$ and called the restriction of $z$ to $A$. We denote by $dim Z$ the covering dimension of $Z$ and by $d(Z)$ its cohomology dimension, that is, $d(Z) = \text{Sup}\{m: H^m(Z) \neq 0\}$. We have $d(Z) < dim Z$ if $Z$ is paracompact. If $q: V \to B$ is a vector space bundle over $B$ then the $j$th Stiefel-Whitney class of $q$ is denoted by $w_j(q)$.

We will assume throughout the paper that $B$ is a paracompact space.

**Theorem.** Let $p: E \to B$ be an $n$-sphere bundle with the antipodal involution, let $q: V \to B$ be an $\mathbb{R}^n$-bundle and let $f: E \to V$ be a fibre preserving map over $B$. If $d(B) < d$ and $w_j(q) = 0$ for $1 < j < r$ then the map $(p|A_f)^*: H^i(B) \to H^i(\tilde{A}_f)$ is injective for $i > d - r$.

In the following corollaries we specify particular cases of this theorem to illustrate its significance.

**Corollary 1.** If $f: E \to V$ is a fibre preserving map of an $n$-sphere bundle $p: E \to B$ with the antipodal involution into an $\mathbb{R}^n$-bundle $q: V \to B$ and if $d(B) = d < \infty$, then the map $(p|A_f)^*: H^d(B) \to H^d(\tilde{A}_f)$ is injective.

**Corollary 2.** If $f: E \to V$ is a fibre preserving map of an $n$-sphere bundle $p: E \to B$ with the antipodal involution into an $\mathbb{R}^n$-bundle $q: V \to B$ and if all the Stiefel-Whitney classes of $q$ are zero then the map $(p|A_f)^*: H^i(B) \to H^i(\tilde{A}_f)$ is injective for every $i$.

**Corollary 3.** If $B$ is closed manifold and $f: E \to V$ is a fibre preserving map of an $n$-sphere bundle $p: E \to B$ with the antipodal involution into an $\mathbb{R}^n$-bundle $q: V \to B$ then $dim A_f = dim \tilde{A}_f > dim B$.

In Corollary 3, we have $d = d(B) = dim B$ and $H^d(B) \neq 0$. On the other hand, $dim \tilde{A}_f = dim A_f$ since the orbit map $A_f \to \tilde{A}_f$ is a double covering.

**Proof of the theorem.** If $X$ is any space with a free involution $t: X \to X$, let $u(X)$ denote its characteristic class. It is an element $u(X) \in H^1(\tilde{X})$, where $\tilde{X}$ is, as usual, the orbit space of $t$. In other words, $u(X)$ is the Stiefel-Whitney class of the double covering $X \to \tilde{X}$. The class $u(S^n)$ of the antipodal involution generates the polynomial ring $H^*(\mathbb{P}^n)$ of height $n$.

Let $b \in B$. Then the fibre of $\tilde{p}$ over $b$ is $\tilde{p}^{-1}b = \mathbb{P}^n$ and the polynomial ring $H^*(\tilde{p}^{-1}b)$ is generated by $u(p^{-1}b) \in H^1(\tilde{p}^{-1}b)$. The fibre inclusion $p^{-1}b \to E$ is an equivariant map. By the naturality of $u$, the restriction of $u(E) \in H^1(\tilde{E})$ to the fibre $\tilde{p}^{-1}b$ is equal to $u(p^{-1}b)$. By the Leray-Dold-Hirsch theorem [3, p. 229], $H^*(\tilde{E})$ is an $H^*(B)$-module freely generated by the powers $1, u(E), \ldots, u^n(E)$, with
$H^*(B)$ acting on $H^*(\overline{E})$ via the cup product. In other words, the map
\[
\bigoplus_{i=0}^{n} H^{m+i}(B) \to H^{m+n}(\overline{E}),
\]
\[(x_m, x_{m+1}, \ldots, x_{m+n}) \mapsto \sum_{i=0}^{n} (\tilde{p}^* x_{m+i}) \cup u^{n-i}(E)\]
is an isomorphism. This map restricted to $H^m(B)$ gives a monomorphism
\[i: H^m(B) \to H^m(\overline{E}), \quad x \mapsto (\tilde{p}^* x) \cup u^m(E)\].

Let $0$ be the zero section in $V$ and $V_0 = V - 0$. Then the antipodal map is a free
involution in $V_0$ and the fibre of the bundle $q_0 = q|V_0$: $V_0 \to B$ is $R^n_0 = R^n - (0)$. The
bundle $q_0$ is fibre homotopy equivalent to its $S^{n-1}$-bundle and hence $H^*(\overline{V}_0)$ is
an $H^*(B)$-module freely generated by $1$, $(\tilde{u}(V_0), \ldots, u^{n-1}(V_0))$. Moreover, $u^n(V_0) = \Sigma_{j=1}^{n} (\tilde{g}_0^* w_j) \cup u^{n-j}(V_0)$, where the coefficient $w_j = w_j(q)$ is the $j$th Stiefel-Whitney
class of $q [3, p. 232]$.

Let $g: E \to V$ be defined by $ge = fe - f(-e)$. Then $g$ is equivariant, $g(-e) = -ge,
A_f = A_g = g^{-1}0$ and the restriction of $g$ to $E_0 = E - A_f$ defines an equivariant
map $g_0: E_0 \to V_0$. By the naturality of $u$, we have $\tilde{g}_0^* u(V_0) = u(E_0)$, where $\tilde{g}_0:
E_0 \to V_0$ is the map of the orbit bundles induced by $g_0$, and $u(E_0) = u(E)|\overline{E}_0$. It
follows that
\[u^n(E)|\overline{E}_0 = \tilde{g}_0^* u^n(V_0) = \sum_{j=1}^{n} [(\tilde{p}^* w_j)|\overline{E}_0] \cup [u^{n-j}(E)|\overline{E}_0]\]
\[= \left[\sum_{j=1}^{n} (\tilde{p}^* w_j) \cup u^{n-j}(E)\right]|\overline{E}_0.

To show that $(\tilde{p}|\overline{A}_f)^* x$ is a monomorphism in the degrees specified in the theorem,
suppose that $x \in H^i(B)$ with $i > d - r$ and $(\tilde{p}|\overline{A}_f)^* x = 0$, i.e., $(\tilde{p}^* x)|\overline{A}_f = 0$. By
the continuity of $H^*$, there is a neighborhood $U$ of $A_f$ in $E$ such that $(\tilde{p}^* x)(U) = 0
(\overline{U}$ denotes, as usual, the image of $U$ in $\overline{E}$). Let $e: E \to (\overline{E}, \overline{U})$ and $k: E \to (\overline{E}, \overline{E}_0)$
be the inclusion maps. Since $(\tilde{p}^* x)(U) = 0$, then $\tilde{p}^* x = e^y$, for some $y \in
H^i(\overline{E}, \overline{U})$. Let $v = u^n(E) - \Sigma_{j=1}^{n-1} (\tilde{p}^* w_j) \cup u^{n-j}(E)$. Then $v|\overline{E}_0 = 0$; hence $v = k^* z$, for some $z \in H^i(\overline{E}, \overline{E}_0)$. Since $(\overline{E}, \overline{U}, \overline{E}_0)$ is an excisive triad, $e^z \cup k^* z = y
\cup z = 0$; hence $0 = (\tilde{p}^* x) \cup u^n(E) - (\tilde{p}^* x) \cup [\Sigma_{j=1}^{n-1} (\tilde{p}^* w_j) \cup u^{n-j}(E)]$. There-

\[(\tilde{p}^* x) \cup u^n(E) = \sum_{j=1}^{n} \tilde{p}^* (x \cup w_j) \cup u^{n-j}(E)\].

Now if $j < r$ then $w_j = 0$ by the assumption. If $j > r$ then $\deg(x \cup w_j) = i + j
> i + r > d > d(B)$ since $i > d - r$. Therefore all the coefficients in this poly-
nomial are zero. Hence $(\tilde{p}^* x) \cup u^n(E) = 0$. But $(\tilde{p}^* x) \cup u^n(E) = \iota x$ and $\iota$ is a
monomorphism. Therefore $x = 0$ and thus $(\tilde{p}|\overline{A}_f)^* x$ is a monomorphism. Q.E.D.

REFERENCES

2. J. E. Connett, On the cohomology of the fixed-point sets and coincidence-point sets, Indiana Univ.

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